Published by Institute of Physics Publishing for SISSA/ISAS

Received: *February 11, 2003*
Revised: *June 9, 2003*
Accepted: *June 10, 2003*

BMN Operators for $\mathcal{N}=1$ superconformal Yang-Mills theories and associated string backgrounds

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Abstract: We study a class of near-BPS operators for a complex 2-parameter family of $\mathcal{N}=1$ superconformal Yang-Mills theories that can be obtained by a Leigh-Strassler deformation of $\mathcal{N}=4$ SYM theory. We identify these operators in the large-N and large R-charge limit and compute their exact scaling dimensions using $\mathcal{N}=1$ superspace methods. From these scaling dimensions we attempt to reverse-engineer the light-cone worldsheet theory that describes string propagation on the Penrose limit of the dual geometry.

Keywords: Conformal Field Models in String Theory, Penrose limit and pp-wave background, AdS-CFT and dS-CFT Correspondence, Superspaces.

JHEP06(2003)015

Contents

1. Introduction

According to the AdS/CFT correspondence [1, 2, 3] string theory on spaces of the form $AdS_d \times \mathcal{M}^{10-d}$ is dual to a conformal field theory that lives on the $(d-1)$-dimensional boundary of AdS_d. Several examples of this correspondence have been studied so far. From the CFT point of view new conformal or nonconformal examples can be obtained by deforming the gauge theory action with a local operator $\mathcal{O}$

$$S \to S + h \int d^d x \mathcal{O}(x) \,. \tag{1.1}$$

Usually such deformations break some or all of the initial supersymmetry and in most cases it is a nontrivial task to determine how this deformation reflects itself on the string theory side. When $\mathcal{O}$ is a relevant operator the deformation breaks conformal invariance and the RG flow can lead to an interacting IR fixed point. On the gravity side such a deformation yields a complicated space with a running dilaton that interpolates between two AdS geometries of the form $AdS \times \mathcal{M}_{UV}$ and $AdS \times \mathcal{M}_{IR}$. When $\mathcal{O}$ is exactly marginal, conformal invariance remains as a true symmetry of the theory and the dual geometry takes the form $AdS \times \mathcal{M}_h$, with $\mathcal{M}_h$ a compact deformed version of the original manifold $\mathcal{M}$.

JHEP06(2003)015

We are interested in type-IIB string theory on $AdS_5 \times S^5$ and deformations of its dual $\mathcal{N}=4$ SYM. Many interesting papers have been written on this subject. For example, non-supersymmetric deformations were discussed in [4]. Exactly marginal and relevant deformations that preserve $\mathcal{N}=1$ supersymmetry were discussed in [5]–[12] (for a brief review see [13], section 4.3). A relevant perturbation that leads to a confining gauge theory was discussed in [14]. All these cases were considered in the large t'Hooft limit, where supergravity is reliable.

Here we want to discuss a certain class of $\mathcal{N}=1$ superconformal Yang-Mills theories that can be obtained by a Leigh-Strassler deformation of the $\mathcal{N}=4$ SYM theory [15, 12]. Our analysis focuses on the properties of various near-BPS gauge theory operators with large R-charge. These operators were considered recently by the authors of [16], who also proposed an exact correspondence between such gauge theory operators and string states on the Penrose limit of the $AdS_5 \times S^5$ geometry. Working solely within the deformed gauge theory we use $\mathcal{N}=1$ superspace methods, in a fashion first proposed in [17], to determine their exact scaling dimensions for any value of the perturbing parameters and at strong t' Hooft coupling. In general, these operators are not protected, since they do not fall into short multiplets of the $\mathrm{SU}(2,2|1)$ superconformal group and they obtain anomalous dimensions as one moves away from the weakly coupled $\mathcal{N}=4$ SYM point. Scaling dimensions of such non-protected operators are expected (already at the $\mathcal{N}=4$ point) to diverge at strong t' Hooft coupling as $(g^2_{\mathrm{YM}}N)^{1/4}$, but as a special property of the large R-charge limit of [16] they approach a finite value at strong t' Hooft coupling.

With these operators at hand and following the spirit of the proposal in [16], we can further ask for a light-cone worldsheet theory, whose spectrum reproduces the scaling dimensions we found. Once the worldsheet theory has been determined, we can further attempt to read off the dual string theory background. We find that such a process does not result in a unique background in the infinite R-charge limit. There is, however, a unique one which exhibits supersymmetry enhancement from sixteen to twenty-four supersymmetries.

This reverse-engineering of a string theory from data available in gauge theory would provide, in general, a very powerful method for uncovering further examples of gauge-gravity duals and one would like to have, if possible, a generic prescription to achieve it. In this paper we use the very special properties of the correspondence proposed in [16]; in order to achieve a similar task in a more generic situation one would first have to understand better how to extend this correspondence to finite R-charge and in cases without conformal invariance and/or no supersymmetry.

For the $\mathcal{N}=4$ SYM theory at large R-charge, we should focus on the Penrose limit of $AdS_5 \times S^5$ [18]–[21]. This limit leads to a maximally supersymmetric background with metric

$$ds^2 = -4dx^+dx^- + \sum_{i=1}^{8} \left(dr^i dr^i - r^i r^i (dx^+)^2\right), \tag{1.2}$$

and constant R-R 5-form flux

$$F_{+1234} = F_{+5678} = \mathrm{const}\,. \tag{1.3}$$

One of the merits of this background is the exact solvability of the associated worldsheet theory in the light-cone Green-Schwarz formalism, where it simply reduces to a sum of massive oscillators [22, 23]. On the gauge theory side the Hilbert space of the $\mathcal{N}=4$ SYM is suitably truncated to states with large scaling dimension $\Delta \sim \sqrt{N}$ and large $\mathrm{U}(1)_R$ R-charge $J \sim \sqrt{N}$, while the difference $(\Delta - J)$ is kept fixed and small. A correspondence between such states and on-shell states of string theory in the bulk pp-wave background was proposed by Berenstein, Maldacena and Nastase (BMN) in [16] and as a check the scaling dimensions on both sides were computed and were found to agree. Further checks of this correspondence (and beyond the planar limit) were performed in [17]–[30].

We can obtain a whole moduli space of $\mathcal{N}=1$ SYM theories by perturbing the $\mathcal{N}=4$ lagrangian by a superpotential that breaks the $\mathrm{SU}(4)_R$ R-symmetry group to a diagonal $\mathrm{U}(1)_R$ under which all six of the Higgs fields are charged. This $\mathrm{U}(1)_R$ is different from the one that was considered in [16] and for that reason it is useful to present a slight variant of that discussion for the $\mathcal{N}=4$ theory. We perform the Penrose limit of $AdS_5 \times S^5$ around the appropriate geodesic and repeat the BMN analysis to rephrase the correspondence between string theory and gauge theory. We find that the resulting pp-wave limit has a metric of the form

$$ds^2 = -4dx^+dx^- + 4\mu y_1 dx_1 dx^+ + 4\mu y_2 dx_2 dx^+ - \mu^2 \vec{r}^2 (dx^+)^2 + d\vec{r}^2 + d\vec{y}^2 + d\vec{x}^2\,, \quad (1.4)$$

and a 5-form field strength of the form

$$F_5 = \mathcal{F}_5 + *\mathcal{F}_5\,, \qquad \mathcal{F}_5 \sim \mu dx^+ \wedge dy^1 \wedge dx^1 \wedge dy^2 \wedge dx^2\,. \quad (1.5)$$

μ is a mass parameter that can be scaled out through the rescaling $x^+ \to x^+/\mu$ and $x^- \to \mu x^-$. In the rest of the paper it is set to one. The Green-Schwarz light-cone worldsheet action includes four massive harmonic oscillators as in [16] and a Landau part that corresponds to the action of a charged particle moving in the presence of a constant magnetic field. This action is again exactly solvable and the string spectrum is known. In fact, after a suitable x^+-dependent change of coordinates the magnetic background of (1.4) transforms into (1.2) [31]. On the gauge theory side, the Penrose limit restricts the $\mathcal{N}=4$ SYM Hilbert space into the same subsector as the one that appears in [16], but the R-charge assignments are now different. As a result, the BMN correspondence involves at each level an infinite degeneracy. On the string theory side this is the usual infinite degeneracy of Landau levels.

The organization of this paper is the following. In section 2, we discuss in detail the Penrose limit of interest and derive the resulting geometry at the $\mathcal{N}=4$ point. We consider string propagation on this geometry and review the associated string spectra. Then, we focus on the gauge theory side and construct the string oscillators from the appropriate gauge invariant SYM operators in the spirit of [16]. This analysis is useful, because it clarifies some characteristics of the BMN correspondence under a different R-charge assignment and it hints as to what may be expected to change or remain the same as we deform away from the $\mathcal{N}=4$ point. In section 3 we briefly review the 2-complex parameter class of exactly marginal deformations of the $\mathcal{N}=4$ SYM theory that will be the main focus of our analysis. This class of theories was introduced in [15] and further studied in connection with AdS/CFT in [6, 9, 10, 11, 12] . We proceed to determine the properties of the BMN operators after the Leigh-Strassler deformations using $\mathcal{N} = 1$ superspace techniques. We write down appropriate two-point functions of these operators and deduce their exact scaling dimensions in a fashion similar to [17]. As a further check of this result, we perform a perturbative calculation to verify in leading order that the scaling dimensions depend on the deforming parameters as expected. In section 4 we use the available gauge theory data to reconstruct the worldsheet action for string propagation in the Penrose limit of the dual geometry and provide a detailed analysis of the supersymmetries preserved by the associated pp-wave. In section 5 we present our conclusions and suggest directions for further research.

2. A "magnetic" pp-wave limit of $AdS_5 \times S^5$ and its gauge theory dual

2.1 The Penrose limit

Let us start with the $AdS_5 \times S^5$ metric

$$ds^2 = R^2\left(-dt^2\cosh^2\rho + d\rho^2 + \sinh^2\rho d\Omega_3^2 + d\psi^2\cos^2\theta + d\theta^2 + \sin^2\theta d\Omega_3'^2\right) \tag{2.1}$$

and write explicitly the solid angle $d\Omega_3'^2$ in S^5 as

$$d\Omega_3'^2 = \cos^2\phi_1 d\phi_2^2 + d\phi_1^2 + \sin^2\phi_1 d\phi_3^2\,. \tag{2.2}$$

In this parametrization, S^5 is given in terms of the five coordinates $(\psi, \theta, \phi_1, \phi_2, \phi_3)$. There are three obvious U(1) isometries and they have to do with translations of the coordinates ψ, ϕ_2 and ϕ_3. On the gauge theory side each of them is in one-to-one correspondence with a $\mathrm{U}(1)_R$ that rotates one of the three complex Higgs fields of the $\mathcal{N} = 4$ theory. We denote them as Φ^1, Φ^2 and Φ^3. We make the correspondence

$$\Phi^1 \leftrightarrow J_{\Phi^1} = -i\partial_\psi\,, \tag{2.3}$$

$$\Phi^2 \leftrightarrow J_{\Phi^2} = -i\partial_{\phi_2}\,, \tag{2.4}$$

$$\Phi^3 \leftrightarrow J_{\Phi^3} = -i\partial_{\phi_3}\,. \tag{2.5}$$

In general, we would like to consider an arbitrary linear superposition of the three U(1) isometries under which the complex fields Φ^1, Φ^2 and Φ^3 have charges Q_1, Q_2 and Q_3 respectively. The Penrose limit will be taken along a null geodesic associated to this isometry. For that purpose we introduce an angular coordinate ω' given by

$$-i\partial_{\omega'} \equiv -i(Q_1\partial_\psi + Q_2\partial_{\phi_2} + Q_3\partial_{\phi_3}) \tag{2.6}$$

and we suitably rescale it to get a new coordinate ω with periodicity 2π. Independently of the charges Q_1, Q_2 and Q_3, we can always write $\omega = \frac{\psi+\phi_2+\phi_3}{3}$ and the charge of every complex Higgs field, as measured by the current $-i\partial_\omega$, is one.

The geodesic of interest is given by

$$t = \omega\,, \qquad \rho = 0\,, \qquad \theta = \theta_0\,, \qquad \phi_1 = \frac{\pi}{4}\,, \qquad \psi = \phi_2 = \phi_3 = \omega\,, \tag{2.7}$$

with $\theta_0 = \arccos(1/\sqrt{3})$. Indeed, a simple substitution of these values in (2.1) gives the null geodesic condition

$$ds^2 = R^2\left(- dt^2 + d\omega^2\right) = 0\,. \tag{2.8}$$

In order to focus on the geometry of the neighborhood of this geodesic we introduce new coordinates

$$x^+ = \frac{1}{2}(t+\omega)\,, \tag{2.9}$$

$$x^- = \frac{R^2}{2}(t-\omega) \tag{2.10}$$

and perform the rescaling

$$\rho = \frac{r}{R}\,, \qquad \theta = \theta_0 + \frac{y_1}{R}\,, \qquad \phi_1 = \frac{\pi}{4} + \sqrt{\frac{3}{2}}\frac{y_2}{R}\,, \qquad \psi = \omega - \sqrt{2}\frac{x_1}{R}\,, \tag{2.11}$$

$$\phi_2 = \omega + \frac{1}{\sqrt{2}}\frac{x_1 - \sqrt{3}x_2}{R}\,, \qquad \phi_3 = \omega + \frac{1}{\sqrt{2}}\frac{x_1 + \sqrt{3}x_2}{R}\,, \tag{2.12}$$

taking the $R \to \infty$ limit. The numerical factors have been inserted for later convenience.

Expanding each expression in (2.1) up to second order in $1/R^2$ gives the pp-wave metric

$$ds^2 = -4dx^+dx^- - r^2(dx^+)^2 + \sum_{i=1}^{4} dr^i dr^i + \sum_{a=1,2}(dy_a^2 + dx_a^2 + 4y_a dx_a dx^+)\,. \tag{2.13}$$

The full solution is also supported by the constant 5-form flux of eq. (1.5). Following [32] we will hereafter refer to this background as the magnetic pp-wave limit of $AdS_5 \times S^5$. It is a maximally supersymmetric background with 32 supersymmetries and its gauge theory dual is a suitable truncation of the $\mathcal{N} = 4$ SYM. This truncation is independent of the choice of the $\mathrm{U}(1)_R$ and therefore it is not different from the one that appears in [16]. It is worth noticing that the same pp-wave background also appears in [31, 32, 33], where the Penrose limit was taken on $AdS^5 \times T^{1,1}$. The gauge theory dual in that case is an $\mathcal{N} = 1$ $\mathrm{SU}(N) \times \mathrm{SU}(N)$ SYM with a pair of bifundamental chiral multiplets A_i and B_i transforming in the $(N, \bar{N})$ and $(\bar{N}, N)$ representation of the gauge group. The fact that it can also be obtained from $AdS_5 \times S^5$ in the fashion that we discuss here was also mentioned in [32].

The correspondence between the light-cone momenta p^- and p^+ on the string theory side and the scaling dimensions and R-charges on the gauge theory side works in the following way

$$2p^- = -p_+ = i\partial_{x^+} = i(\partial_t + \partial_\omega) = \Delta - J \tag{2.14}$$

and

$$2p^+ = -p_- = i\partial_{x^-} = \frac{1}{R^2}i(\partial_t - \partial_\omega) = \frac{1}{R^2}(\Delta + J)\,. \tag{2.15}$$

R is the radius of AdS_5 and we have set

$$J = -i\partial_\omega = \frac{1}{Q_1}R_1 + \frac{1}{Q_2}R_2 + \frac{1}{Q_3}R_3\,. \tag{2.16}$$

For each $i = 1, 2, 3$, R_i is a $\mathrm{U}(1)$ generator under which only Φ^i is charged and the charge is Q_i.

JHEP06(2003)015

In the limit under consideration $R \to \infty$. Since we only keep the states with finite p^+ it is necessary to take the familiar scaling $\Delta, J \sim R^2 \sim \sqrt{N}$. As a result, on the gauge theory side we must take the $N \to \infty$ limit keeping the Yang-Mills coupling fixed and small and focus on operators with large R-charge $J \sim \sqrt{N}$ and small and fixed $\Delta - J$. Such operators were introduced in [16] and we re-discuss them in the magnetic pp-wave context in section 2.3.

2.2 String propagation on magnetic pp-waves

The gauge-fixed light-cone bosonic string action for the background (2.13) is [32]

$$S = \frac{1}{2\pi\alpha'} \int d\tau \int_0^{2\pi\alpha' p^+} d\sigma \left(\frac{1}{2}\partial_a \vec{r} \partial^a \vec{r} - \frac{1}{2} r^2 + \frac{1}{2}\partial_a \vec{x} \partial^a \vec{x} + \frac{1}{2}\partial_a \vec{y} \partial^a \vec{y} - 2\vec{x}\partial_\tau \vec{y} \right) . \quad (2.17)$$

There are several terms contributing to this action. There are four massive oscillators labeled by the 4-dimensional vector $\vec{r}$ and two identical decoupled Landau actions involving the coordinates (x_1, y_1) and (x_2, y_2). Each of them is precisely the action of a 2-dimensional charged particle moving in a constant magnetic field. It is convenient to rewrite the $x - y$ part of the action by performing the rotation

$$x_a = -\frac{1}{\sqrt{2}}(\hat{x}_a + \hat{y}_a) , \qquad y_a = \frac{1}{\sqrt{2}}(\hat{x}_a - \hat{y}_a) . \quad (2.18)$$

Up to a total derivative term that can be dropped the action takes the form

$$S_{xy} = \frac{1}{2\pi\alpha'} \int d\tau \int_0^{2\pi\alpha' p^+} d\sigma \left(\frac{1}{2}\partial_a \vec{\hat{x}} \cdot \partial^a \vec{\hat{x}} + \frac{1}{2}\partial_a \vec{\hat{y}} \cdot \partial^a \vec{\hat{y}} - \vec{\hat{x}} \cdot \partial_\tau \vec{\hat{y}} + \vec{\hat{y}} \cdot \partial_\tau \vec{\hat{x}} \right) \quad (2.19)$$

and from now on we drop the $\hat{}$ notation. This action and the associated spectrum have also appeared in the context of the Penrose limit of $AdS_5 \times T^{1,1}$ in [31]. For completeness, in the rest of this subsection we review the spectra that were obtained there.

The spectrum of the r^i part of the light-cone hamiltonian reads

$$\mathcal{H}_r = \sum_{n=-\infty}^{\infty} N_n^{(r)} \sqrt{1 + \left(\frac{n}{\alpha' p^+}\right)^2} . \quad (2.20)$$

There are four kinds of oscillators contributing to the level $N_n^{(r)}$ and we denote them as a_n^i, for $i = 1, 2, 3, 4$. We use the notation of [16], so $n > 0$ label the left movers and $n < 0$ label the right movers.

For the $x - y$ part of the action the light-cone hamiltonian breaks up into four parts

$$\mathcal{H}_{xy} = \sum_{n=-\infty}^{\infty} \sum_{a=1,2} \left[N_n^{(b^a)} \left(\sqrt{1 + \left(\frac{n}{\alpha' p^+}\right)^2} + 1 \right) + N_n^{(\bar{b}^a)} \left(\sqrt{1 + \left(\frac{n}{\alpha' p^+}\right)^2} - 1 \right) \right] . \quad (2.21)$$

Four types of oscillators contribute to each of the above terms. The oscillators $(b_n^1, \bar{b}_n^1)$ originate from the (x_1, y_1) part of the lagrangian and contribute to the levels $N_n^{b^1}$ and $N_n^{\bar{b}^1}$ respectively and the oscillators $(b_n^2, \bar{b}_n^2)$ contribute to the levels $N_n^{b^2}$ and $N_n^{\bar{b}^2}$.

JHEP06(2003)015

These spectra can be derived by straightforward calculation, or they can be deduced from the following slightly different point of view [31]. After the change of variables (2.18), we introduce the complex coordinates $z_a = x_a + iy_a$ and we bring the metric (2.13) into the form

$$ds^2 = -4dx^+dx^- - r^2(dx^+)^2 + \sum_{i=1}^{4} dr^i dr^i + \sum_{a=1,2} \left(dz_a d\bar{z}_a + i(\bar{z}_a dz_a - z_a d\bar{z}_a)dx^+\right) . \quad (2.22)$$

This background can be transformed into the maximally supersymmetric pp-wave solution of [16] if we perform the x^+-coordinate dependent $\mathrm{U}(1) \times \mathrm{U}(1)$ rotation

$$z_a = e^{ix^+} w_a \,, \qquad \bar{z}_a = e^{-ix^+} \bar{w}_a \,. \quad (2.23)$$

In view of (2.14) this translates to

$$\Delta - J = i\partial_{x^+}|_{z_a} = i\partial_{x^+}|_{w_a} + \sum_a (w_a \partial_{w_a} - \bar{w}_a \partial_{\bar{w}_a}) = (\Delta - J)_{S^5} + J_1 + J_2 \,, \quad (2.24)$$

where J_1 and J_2 are $\mathrm{U}(1)$ rotation charges in the $(w_1, \bar{w}_1)$ and $(w_2, \bar{w}_2)$ transverse planes respectively.

The spectra of eqs. (2.20), (2.21) can be reproduced from (2.24) by noticing that the bosonic oscillators have the following J_1, J_2 charges

$$\begin{array}{lll} a_n^i & J_1 = J_2 = 0 \,, & i = 1,2,3,4 \\ b_n^1 & J_1 = 1 \,, & J_2 = 0 \,, \\ \bar{b}_n^1 & J_1 = -1 \,, & J_2 = 0 \,, \\ b_n^2 & J_1 = 0 \,, & J_2 = 1 \,, \\ \bar{b}_n^2 & J_1 = 0 \,, & J_2 = -1 \,. \end{array} \quad (2.25)$$

The fermionic oscillator contributions to the light-cone hamiltonian p^- can be similarly deduced from (2.24) by looking at the $\mathrm{U}(1) \times \mathrm{U}(1)$ charges carried by the $\mathrm{SO}(8)$ spinor $\mathbf{8}_s$ under the $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1) \times \mathrm{U}(1)$ into which $\mathrm{SO}(8)$ has been broken [31]

$$\mathbf{8}_s \to (\mathbf{2},\mathbf{1})_{(1/2,1/2)} \oplus (\mathbf{2},\mathbf{1})_{(-1/2,-1/2)} \oplus (\mathbf{1},\mathbf{2})_{(1/2,-1/2)} \oplus (\mathbf{1},\mathbf{2})_{(-1/2,1/2)} \,. \quad (2.26)$$

We get the spectra

$$\begin{array}{lll} S_n^{\alpha++} & 2p^- = & \sqrt{1 + \left(\frac{n}{\alpha' p^+}\right)^2} + 1 \,, \\ S_n^{\alpha--} & 2p^- = & \sqrt{1 + \left(\frac{n}{\alpha' p^+}\right)^2} - 1 \,, \\ S_n^{\dot{\alpha}+-} & 2p^- = & \sqrt{1 + \left(\frac{n}{\alpha' p^+}\right)^2} \,, \\ S_n^{\dot{\alpha}-+} & 2p^- = & \sqrt{1 + \left(\frac{n}{\alpha' p^+}\right)^2} \,, \end{array} \quad (2.27)$$

which, as expected, turn out to be identical to the bosonic ones.

Notice that the action of the bosonic zero mode oscillators $\bar{b}_0^1$ and $\bar{b}_0^2$, as well as the action of their fermionic superpartners $S_0^{\alpha--}$ has no effect on the light-cone energy. As a result, the spectrum exhibits an infinite degeneracy. The degenerate states are obtained by the action of an arbitrary number of the above zero mode oscillators on the vacuum. This degeneracy is familiar, since the worldsheet action contains two decoupled Landau parts, which describe a charged particle moving in the presence of a constant magnetic field in $\mathbf{R}^2 \times \mathbf{R}^2$. This system is known to have an infinite degeneracy of states labeled by the angular momentum of the charged particle.

In the next section we discuss how these bulk characteristics manifest themselves on the dual gauge theory.

2.3 The gauge/string correspondence

Now we would like to discuss the correspondence between the string oscillator states of the previous section and appropriate operators in the dual $\mathcal{N}=4$ SYM theory. Following [16] we are interested in the large-N limit with $g_{\rm YM}^2$ kept fixed and small. We work in the planar limit and examine single trace operators, which we categorize by their $\Delta - J$ value. As in the usual BMN limit there exists a very interesting finite J version of these operators [34, 30], which we do not discuss in this paper.

We begin with single trace operators of $\Delta - J = 0$. There is an infinite number. Any traceless operator of the form $\mathrm{Tr}[\Phi^1 \cdots \Phi^2 \cdots \Phi^3 \cdots]$ containing J symmetrized insertions of the Φ^1, Φ^2 or Φ^3 fields has $\Delta - J = 0$. Each of them is an $\mathcal{N}=4$ chiral primary and its scaling dimension is protected by supersymmetry.

In order to construct the correspondence of SYM operators with string oscillator states, it is perhaps natural to single out a specific linear superposition of the Higgs fields associated to the $\mathrm{U}(1)_R$ generator J that appears in (2.16). We choose the diagonal superposition

$$\Omega = \frac{1}{\sqrt{3}}(\Phi^1 + \Phi^2 + \Phi^3)\,. \tag{2.28}$$

In the language of [16] we propose the correspondence

$$\frac{1}{\sqrt{J}N^{J/2}}\,\mathrm{Tr}[\Omega^J] \leftrightarrow |0, p^+; \sigma_\Omega\rangle_{l.c.}\,, \tag{2.29}$$

where σ_Ω is a formal parameter that denotes a particular state of the infinitely degenerate light-cone vacuum space. The factor of the l.h.s. is such that the normalization of the two point function is one.

To obtain the rest of the $\Delta - J = 0$ operators we act on the above vacuum with an arbitrary number of the zero mode oscillators $\bar{b}_0^1, \bar{b}_0^2$. Since they have no effect on the light-cone energy, these oscillators should be associated again to linear combinations of the Higgs fields Φ^1, Φ^2 and Φ^3. We choose the two linear combinations that are orthogonal to Ω and propose the correspondence

$$\bar{b}_0^1 \leftrightarrow \Psi^1 = \frac{1}{\sqrt{2}}(\Phi^2 + \Phi^3 - 2\Phi^1) \tag{2.30}$$

and

$$\bar{b}_0^2 \leftrightarrow \Psi^2 = \frac{1}{\sqrt{6}}(\Phi^3 - \Phi^2)\,. \tag{2.31}$$

It is clear that the above correspondence between operator insertions and string oscillators is by no means unique. Any SU(3) rotated basis of Higgs fields could equally well be assigned to the same string oscillators. This lack of uniqueness is also manifest on the arbitrary choice of the state $|0,p^+;\sigma_\Omega\rangle_{l.c.}$ on the r.h.s. of (2.29).

With the above correspondence the action of the zero mode oscillators $\bar{b}_0^{a\dagger}$ $(a=1,2)$ on the light-cone vacuum (2.29) can be translated in the SYM language as follows. For each $\bar{b}_0^{a\dagger}$ we are instructed to make an insertion of Ψ^a and then sum over all possible orderings. This is the same as acting on $\mathrm{Tr}[\Omega^J]$ with the operator $\sum_{l=1}^{J}(\Omega\Psi^a\frac{\partial}{\partial\Omega})_l$, where we use the notation $(\ldots)_l$ to denote that the operator in parenthesis acts on the lth insertion of the trace. For example,

$$\frac{1}{\sqrt{J}}\sum_l \frac{1}{\sqrt{J}N^{J/2+1/2}}\mathrm{Tr}\left[\Omega^l\Psi^a\Omega^{J-l-1}\right] \leftrightarrow \bar{b}_0^{a\dagger}|0,p^+;\sigma_\Omega\rangle_{l.c.}\,. \tag{2.32}$$

Repeated action of these zero modes creates the anticipated Landau degeneracy of the vacuum, which becomes infinite in the $J\to\infty$ limit.

For the operators with $\Delta - J = 1$ we can say the following. There are *twelve* bosonic operators of this type, $D_i\Omega$, $D_i\Psi^1$ and $D_i\Psi^2$ and they are expected to match the four zero mode oscillators a_0^i for $i=1,2,3,4$. This correspondence works by associating

$$a^{i\dagger} \leftrightarrow \sum_l (\Omega D_i)_l \quad i=1,2,3,4\,. \tag{2.33}$$

D_i denotes the gauge covariant derivative with respect to the spacetime coordinates of $\mathbf{R}^4$ where the dual $\mathcal{N}=4$ gauge theory lives. More precisely, whenever we act on the vacuum $|0,p^+;\sigma_\Omega\rangle_{l.c.}$ of eq. (2.29) with the oscillator $a^{i\dagger}$, we are instructed to add an insertion of $D_i\Omega$ on the gauge theory operator $\mathrm{Tr}[\Omega^J]$ and then sum over all possible orderings, e.g.

$$\frac{1}{\sqrt{J}}\sum_{l=0}^{J}\frac{1}{\sqrt{J}N^{J/2+1/2}}\mathrm{Tr}[\Omega^l D_i\Omega\Omega^{J-l}] \leftrightarrow a_0^{i\dagger}|0,p^+;\sigma_\Omega\rangle_{l.c.}\,. \tag{2.34}$$

Acting on a different vacuum state of the same light-cone energy, e.g. acting on $\bar{b}_0^{a\dagger}|0,p^+;\sigma_\Omega\rangle_{l.c.}$, also amounts to a similar insertion of $D_i\Omega$ or $D_i\Psi^a$. We insert $D_i\Omega$ if a position is initially occupied by Ω and $D_i\Psi^a$ if the position is initially occupied by Ψ^a. This rule is a consequence of the fact that the state $a_0^{i\dagger}\bar{b}_0^{a\dagger}|0,p^+;\sigma_\Omega\rangle_{l.c.}$ can also be written as $\bar{b}_0^{a\dagger}a_0^{i\dagger}|0,p^+;\sigma_\Omega\rangle_{l.c.}$.

Finally, we have to consider insertions of the $\Delta - J = 2$ operator $\bar{\Psi}^a$. From the string spectrum (2.21) it is apparent that such insertions correspond to the action of the zero mode oscillators $b_0^{a\dagger}$, which increase the light-cone hamiltonian by 2. It is therefore natural to make the identification

$$\frac{1}{\sqrt{J}}\sum_l \frac{1}{\sqrt{J}N^{J/2+1/2}}\mathrm{Tr}\left[\Omega^l(\bar{\Psi}^a)\Omega^{J-l}\right] \leftrightarrow b_0^{a\dagger}|0,p^+;\sigma_\Omega\rangle_{l.c.}\,. \tag{2.35}$$

JHEP06(2003)015

The above correspondence also extends nicely to the fermionic zero mode oscillators (2.27). The relevant SYM operators follow easily from the bosonic ones by supersymmetry. We have

gauge theory fermionic operators $\leftrightarrow$	**fermionic string oscillators**
$\bar{\lambda}^{\dot{\alpha}+-}$	$S^{\dot{\alpha}+-}$,
$\bar{\lambda}^{\dot{\alpha}-+}$	$S^{\dot{\alpha}-+}$,
$\bar{\psi}^{1}$	S^{1++} ,
$\bar{\psi}^{2}$	S^{2++} ,
ψ^{1}	S^{1--} ,
ψ^{2}	S^{2--} . (2.36)

$\bar{\lambda}$ denotes the right-handed gauginos. There are 8 such components. Each of them has a definite charge ($\pm 1/2$) under the two "Landau" U(1)'s into which $SO(4) \subset SO(6)_R$ has been broken. The $+/-$ superscripts denote the components of the gauginos with charges $\pm 1/2$ respectively. ψ^a for $a = 1, 2$ are the fermionic superpartners of the bosons Ψ^a.

For the higher excited modes of the string the correspondence works exactly as in [16]. The action of any excited oscillator is expressed in the SYM language by the insertion of the corresponding field multiplied by a position dependent phase, e.g.

$$\frac{1}{\sqrt{J}} \sum_{l} \frac{1}{N^{J/2+1}} \mathrm{Tr}[\Psi^a \Omega^l \Psi^b \Omega^{J-l}] e^{\frac{2\pi i n l}{J}} \leftrightarrow \bar{b}_n^{b\dagger} \bar{b}_{-n}^{a\dagger} |0, p^+; \sigma_\Omega\rangle_{l.c.} \,. \tag{2.37}$$

The details of this construction are precisely the same as in [16] and we will not discuss them further.

In conclusion, we rephrased the BMN correspondence at the $\mathcal{N} = 4$ SYM fixed line for a diagonal $U(1)_R$ choice. We did not go into much detail, because the essence of the correspondence is expected to be independent of this choice and in particular, it should be easy to translate all the checks and extensions of the correspondence at finite J in the language of this section. Furthermore, it is natural to expect that this same BMN correspondence also persists when we deform away from the $\mathcal{N} = 4$ fixed line. The goal of the next section is to determine the effect of the deformation on the BMN operators.

3. $\mathcal{N} = 1$ superconformal theories and BMN operators

3.1 Exactly marginal deformations of the $\mathcal{N} = 4$ SYM theory

After this long parenthesis on magnetic pp-waves, we are now ready to proceed with the analysis of the Leigh-Strassler deformations of the $\mathcal{N} = 4$ SYM theory. The four-dimensional $\mathcal{N} = 4$ SU(N) SYM theory can be expressed in the language of $\mathcal{N} = 1$ supersymmetry in terms of a vector multiplet V and three chiral multiplets[1] Φ^i, $i = 1, 2, 3$. In addition to the usual kinetic terms of the $\mathcal{N} = 1$ theory one is also instructed to add a superpotential of the form

$$W = g' \mathrm{Tr}\left([\Phi^1, \Phi^2]\Phi^3\right) . \tag{3.1}$$

[1]In this section Φ^i, Ω and Ψ^a denote full $\mathcal{N} = 1$ superfields and they should not be confused with the bosonic bottom components of the previous section.

JHEP06(2003)015

In this $\mathcal{N} = 1$ language only an $\mathrm{SU}(3) \times \mathrm{U}(1)$ subgroup of the full $\mathrm{SU}(4)_R$ R-symmetry group is manifest. SU(3) is the group that rotates the chiral superfields Φ^i. At the $\mathcal{N} = 4$ point the superpotential coupling g' is directly related to the Yang-Mills coupling and in our conventions $g' = \sqrt{2} g_{\mathrm{YM}}$. To set our notation straight we write the full $\mathcal{N} = 4$ action as

$$\mathcal{S} = \mathrm{Tr}\left(\int d^4\theta e^{-gV} \bar{\Phi}_i e^{gV} \Phi^i + \frac{1}{2g^2} \left[\int d^4x d^2\theta W^\alpha W_\alpha + \int d^4x d^2\bar{\theta} \bar{W}^{\dot{\alpha}} \bar{W}_{\dot{\alpha}} \right] + \right. \\ \left. + \frac{g'}{3!} \int d^4x d^2\theta \epsilon_{ijk} \Phi^i [\Phi^j, \Phi^k] - \frac{g'}{3!} \int d^4x d^2\bar{\theta} \epsilon^{ijk} \bar{\Phi}_i [\bar{\Phi}_j, \bar{\Phi}_k] \right) \tag{3.2}$$

and by definition we always set $g = \sqrt{2} g_{\mathrm{YM}}$. Notice the explicit distinction between the superpotential coupling g' and the vector superfield coupling g. At the $\mathcal{N} = 4$ fixed line we have $g = g'$ but this relation is modified as we deform away and in general we need to differentiate between the two couplings.

Since the $\mathcal{N} = 4$ theory is conformal for any value of the complex coupling $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{\mathrm{YM}}^2}$, the deformation that changes this value is obviously exactly marginal. It is also known, however, that for $N \geq 3$ the $\mathcal{N} = 4$ theory has additional exactly marginal perturbations [15]. Classically, one possibility is given by the superpotential

$$W = h_{ijk} \mathrm{Tr}(\Phi^i \Phi^j \Phi^k) \,, \tag{3.3}$$

with ten symmetric coefficients h_{ijk}. Another one is the superpotential (3.1) with any (complex) coefficient g'. For the first class, it is known [12, 13, 15] that only a two-complex parameter subset of them is exactly marginal on the quantum level. The resulting superpotential can be written as

$$W_{def} = h_1 \mathrm{Tr}\left(\Phi^1\Phi^2\Phi^3 + \Phi^1\Phi^3\Phi^2\right) + h_2 \mathrm{Tr}\left((\Phi^1)^3 + (\Phi^2)^3 + (\Phi^3)^3 \right), \tag{3.4}$$

in terms of two complex coefficients h_1, h_2. These particular deformations preserve a $Z_3 \times Z_3$ symmetry given by the transformations $\Phi^1 \to \Phi^2$, $\Phi^2 \to \Phi^3$, $\Phi^3 \to \Phi^1$ and $\Phi^1 \to \Phi^1$, $\Phi^2 \to \omega\Phi^2$, $\Phi^3 \to \omega^2\Phi^3$. ω is a cubic root of unity. The second Z_3 prevents any mixing between the chiral operators Φ^i and the first can be used to show that they all have the same anomalous dimension $\gamma(\tau, g', h_1, h_2)$. The beta functions are restricted by non-renormalization theorems to be proportional to this anomalous dimension and the constraint

$$\gamma(g, g', h_1, h_2) = 0 \tag{3.5}$$

gives a 3-complex dimensional surface of fixed points. For simplicity, we set the theta angle to zero.

The analytic form of this surface is only known up to first order in perturbation theory [12, 35, 36]. Notice that for generic points in this moduli space the coefficient g' is not necessarily equal to the $\mathcal{N} = 4$ value $g = \sqrt{2} g_{\mathrm{YM}}$. It turns out that the large R-charge limit, on which we base our analysis, probes a neighborhood of this moduli space around the strong 't-Hooft coupling point. Thus, for later considerations it is convenient to write g' as $g' = g + h_0$, with h_0 complex. At the end of the day, our results on the anomalous dimensions of the BMN operators will be expressed in terms of the three independent couplings g, h_1 and h_2.

JHEP06(2003)015

The conclusion of this short introduction is that for fixed g there are basically two exactly marginal deformations away from the $\mathcal{N}=4$ fixed line and they correspond to the superpotential (3.4). On the supergravity side this deformation can be identified at first order with part of the KK scalar mode in the **45** of SO(6) [6, 37]. This scalar corresponds to the second two-form harmonic $Y^I_{[\alpha,\beta]}$ in the expansion of the complex antisymmetric two-form $A_{\alpha,\beta}$ with components along the five-sphere. The effect of the deformation in supergravity has been analyzed perturbatively in the deformation parameters in [6, 12] and is expected to be a warped fibration of AdS_5 over a deformed $\tilde{S}^5$ in the presence of 3-form and 5-form fluxes. An interesting class of supergravity solutions of this type was also obtained in [11]. These solutions, however, appear to be singular and their exact relation to the deformation superpotential (3.4) is not clear.

3.2 BPS and near-BPS operators

In section 2 and in the context of a "magnetic" Penrose limit of $AdS_5 \times S^5$ we considered a class of large R-charge operators of the $\mathcal{N}=4$ SYM theory, which were obtained from the operator

$$\Pi_J \equiv \frac{1}{\sqrt{J}N^{J/2}}\,\mathrm{Tr}[\Omega^J] \tag{3.6}$$

by insertions of the fields $D_i\Omega$, Ψ^a and $\bar{\Psi}^a$ ($i=1,2,3,4$ and $a=1,2$) with or without position dependent phases. Without such phases the resulting symmetrized operators are 1/2-BPS. They are protected operators of the $\mathcal{N}=4$ theory because they belong to short multiplets of the $SU(2,2|4)$ superconformal group.[2]

Alternatively, we can ask in what sense they are protected from an $\mathcal{N}=1$ point of view. Generically an $\mathcal{N}=4$ short multiplet can break into $\mathcal{N}=1$ short and long multiplets and it is not immediately obvious how the $\mathcal{N}=4$ protection manifests itself in the $\mathcal{N}=1$ formalism. This question is even more important and instructive in anticipation of the Leigh-Strassler deformation that breaks the $\mathcal{N}=4$ supersymmetry down to $\mathcal{N}=1$. We need to know what remains protected even after the deformation. The $\mathcal{N}=1$ of interest is the one that is preserved by the Leigh-Strassler deformations, i.e. one under which all three Higgs fields have equal R-charge 2/3.

Let us first see what happens along the $\mathcal{N}=4$ fixed line from an $\mathcal{N}=1$ point of view. Π_J is protected, because it is an $\mathcal{N}=1$ chiral primary operator and obeys the BPS condition $\Delta = J$. The same is also true for the operators that arise when we include symmetrized insertions of the fields Ψ^a. Insertions of the fields $D_i\Omega$ lead to descendants of Π_J and they are also protected. The remaining operators are those with $\bar{\Psi}^a$ insertions. Every such insertion has $\Delta - J = 2$ at weak coupling and clearly does not produce an $\mathcal{N}=1$ chiral field. Nevertheless, the resulting operator is still $\mathcal{N}=1$ protected, because it belongs to another type of short multiplet of $SU(2,2|1)$ and in $\mathcal{N}=1$ notation it is known as a semi-conserved superfield (see, for example, [39]). Semi-conserved superfields L obey

[2] More specifically, they are protected because they belong to short multiplets that cannot combine to form long multiplets after the $\mathcal{N}=4$ interaction is turned on. See e.g. [38] for a recent discussion on this point.

the condition[3]

$$\bar{D}^2 L = 0\,. \tag{3.7}$$

Using the $\mathcal{N} = 4$ SYM equations of motion one can easily verify that the corresponding superfields with $\bar{\Psi}^a$ insertions indeed satisfy this condition.

On the other hand, operators with the above insertions and position-dependent phases are not protected, because the insertions are not symmetrized. For example, operators of the type

$$\sum_l e^{\frac{2\pi i n}{J}}\, \mathrm{Tr}[\Psi^a \Omega^l \Psi^b \Omega^{J-l}] \tag{3.8}$$

have $\Delta - J = 0$ at weak coupling and they may seem to be chiral and hence protected. This, however, is not correct, because one can use the $\mathcal{N} = 4$ SYM equations of motion to symmetrize this operator. In the process extra terms appear and they turn out to be descendants of non-protected operators. A similar reasoning can also be applied to other non-symmetrized operators.

Once we deform the $\mathcal{N} = 4$ SYM action by the superpotential (3.4) at a generic point of the moduli manifold (3.5) many of the above statements about the 0-level BPS operators change. As we verify explicitly in the next section, the deformation modifies the $\mathcal{N} = 4$ equations of motion and the previously protected operators acquire nonzero anomalous dimensions. For example, it is easy to check that (3.7) breaks down away from the $\mathcal{N} = 4$ point and operators with $\bar{\Psi}^a$ insertions no longer remain semi-conserved in the deformed theories. Similarly, the previously symmetrized chiral operators with Ψ^a insertions acquire anomalous dimensions and they are not protected against the $\mathcal{N} = 4$-breaking deformations. These anomalous dimensions are computed in the next section using the technology of [17] and they are verified independently to leading order in perturbation theory in section 3.3.

Only one operator remains protected and continues to have $\Delta - J = 0$. This is Π_J. The vanishing of its anomalous dimension is synonymous to the condition (3.5) that guarantees the presence of superconformal invariance in the deformed theory. As a result, we see that the effect of the deformation is to lift the infinite Landau degeneracy of the $\mathcal{N} = 4$ point and retain a single vacuum state represented on the gauge theory side by the operator Π_J. Such a vacuum state with vanishing light-cone energy should also be expected from the supersymmetry of the dual background.

Another aspect of this picture is the following. We have concentrated our attention on the BMN operators that can be obtained from Π_J by appropriate insertions of other fields and worked mainly in a "dilute gas" approximation. In doing so, we break the Z_3 symmetry that permutes the three adjoint chiral superfields and the "vacuum" operators $\mathrm{Tr}[(\Psi^1)^J]$ and $\mathrm{Tr}[(\Psi^2)^J]$ remain at "infinite distance" from the operator Π_J, i.e. they result from infinite insertions. This seems to be inconsequential for the BMN correspondence at the $\mathcal{N} = 4$ point, because of the infinite Landau degeneracy, but it is perhaps a little puzzling for the BMN correspondence after the deformation. These operators have similar

[3] D_α and $\bar{D}_{\dot\alpha}$ are the usual superspace covariant derivatives. In what follows, we work in $\mathcal{N} = 1$ superspace and adopt the notations of [40].

properties as Π_J and they continue to have $\Delta - J = 0$ throughout the moduli space. In order to obtain them from Π_J we have to start adding insertions that increase the total $\Delta - J$[4] and it is not completely obvious how we can recover an operator with $\Delta - J = 0$. The key point has to be that after several insertions the "dilute gas" approximation starts breaking down and one has to be more careful on the derivation of the scaling dimensions. This process is also obscured by the fact that we have to add an infinite number of insertions and this is not something completely well-defined.

3.3 Exact scaling dimensions in superspace formalism

In order to calculate the anomalous dimensions of the above operators, we would like to determine the appropriate two-point functions. The authors of [17] performed a similar calculation at the $\mathcal{N} = 4$ point by working in superspace formalism and using the constraint imposed by the equations of motion of the theory.[5] Following their example, we consider the operators[6]

$$\mathcal{U}_J^a = \sum_l e^{il\varphi}\Omega^l \bar{\Psi}^a \Omega^{J-l} \tag{3.9}$$

and

$$\mathcal{O}_J^a = \sum_l e^{il\varphi}\Omega^l \Psi^a \Omega^{J-l}\,, \tag{3.10}$$

for $a = 1, 2$ and $\varphi = \frac{2\pi n}{J}$. The actual operators that appear in the BMN construction are traced gauge invariant operators of the type

$$\sum_l e^{il\varphi}\,\mathrm{Tr}[\Psi^a \Omega^l \Psi^b \Omega^{J-l}]\,. \tag{3.11}$$

They contain the above $\mathcal{U}_J^a$ and $\mathcal{O}_J^a$ as "building blocks" and under the "dilute gas" approximation the latter are the dominant pieces in the calculation of the anomalous dimensions.

In the presence of the deformations the gauge theory equations of motion become

$$\begin{aligned} \frac{1}{4}\bar{D}^2\bar{\Psi}^1 &= g'[\Psi^2, \Omega] + h_1\{\Psi^2, \Omega\} + 3h_2(\Psi^1)^2\,, \\ \frac{1}{4}\bar{D}^2\bar{\Psi}^2 &= -g'[\Psi^1, \Omega] + h_1\{\Psi^1, \Omega\} + 3h_2(\Psi^2)^2\,. \end{aligned} \tag{3.12}$$

Notice that the gauge theory action has been expressed in terms of the rotated basis of superfields (Ω, Ψ^1, Ψ^2). This is not necessary, but we do it here in order to comply with the conventions adopted in section 2.3. In the large J limit the above equations imply

$$\begin{aligned} \frac{1}{4}\bar{D}^2\mathcal{U}_J^1 &= (g'(1 - e^{-i\varphi}) + h_1(1 + e^{-i\varphi}))\mathcal{O}_{J+1}^2 + 3h_2\mathcal{O}_J^{11}\,, \\ \frac{1}{4}\bar{D}^2\mathcal{U}_J^2 &= (-g'(1 - e^{-i\varphi}) + h_1(1 + e^{-i\varphi}))\mathcal{O}_{J+1}^1 + 3h_2\mathcal{O}_J^{22}\,, \end{aligned} \tag{3.13}$$

[4] For the type of $\Delta - J$ values that we find after the deformation, see for example table 1.

[5] A similar calculation for $\mathcal{N} = 2$ superconformal gauge theories based on ADE quiver diagrams was performed in [41].

[6] Gauge invariance demands that the operator $\mathcal{U}_J^a$ should be written as $\sum_l e^{il\varphi}\Omega^l e^{-gV}\bar{\Psi}^a e^{gV}\Omega^{J-l}$. For our purposes, however, it is enough to work with the assumption that $V = 0$.

where we have denoted

$$\mathcal{O}_J^{aa} = \sum_l e^{il\varphi}\Omega^l(\Psi^a)^2\Omega^{J-l}\,, \tag{3.14}$$

for $a = 1, 2$. An immediate consequence is the following relation

$$\frac{1}{16}\langle \bar{D}^2\mathcal{U}_J^1(z)D^2\bar{\mathcal{U}}_J^1(z')\rangle = |A_1|^2\langle\mathcal{O}_{J+1}^2(z)\bar{\mathcal{O}}_{J+1}^2(z')\rangle + 9|h_2|^2\langle\mathcal{O}_J^{11}(z)\bar{\mathcal{O}}_J^{11}(z')\rangle + \\ +3A_1\bar{h}_2\langle\mathcal{O}_{J+1}^2(z)\bar{\mathcal{O}}_J^{11}(z')\rangle + 3\bar{A}_1 h_2\langle\mathcal{O}_J^{11}(z)\bar{\mathcal{O}}_{J+1}^2(z')\rangle\,. \tag{3.15}$$

We have defined

$$A_1 = g'(1-e^{-i\varphi}) + h_1(1+e^{-i\varphi}) \tag{3.16}$$

and $z = (x, \theta, \bar{\theta})$ is a superspace variable. There is a similar expression for the two-point function $\langle \bar{D}^2\mathcal{U}_J^2(z)D^2\bar{\mathcal{U}}_J^2(z')\rangle$ with the factor A_1 replaced by

$$A_2 = -g'(1-e^{-i\varphi}) + h_1(1+e^{-i\varphi})\,. \tag{3.17}$$

The aim is to write down an explicit expression for each side of equation (3.15) and use it to deduce a constraint on the anomalous dimensions of interest. Each side can be written down explicitly for any value of the couplings g, g', h_1 and h_2, as long as these couplings obey the constraint (3.5) and as long as every operator that appears in (3.13) is quasi-primary. We will assume that the second condition is valid throughout our calculation even though we have not been able to find an explicit proof. Note that the same assumption was also made in the $\mathcal{N} = 4$ case [16, 17]. This is a crucial ingredient of this approach and it would be worthwhile to investigate it further. We expect it to be valid in the infinite J limit on the basis of the gauge theory/pp-wave string correspondence described in section 2.3.

Now, the key point of the computation is that superconformal invariance determines the form of two-point functions of quasi-primary operators uniquely (up to a normalization-dependent factor). To see how the two-point functions look like we start with a few simple expressions at tree-level. For a generic chiral superfield Φ we have

$$\langle\Phi(z)\bar{\Phi}(z')\rangle = \frac{1}{16}\frac{1}{4\pi^2}\bar{D}^2D^2\frac{\delta^4(\theta-\theta')}{|x-x'|^2}\,. \tag{3.18}$$

Using Wick's theorem we can further show that

$$\begin{aligned}\langle\mathcal{O}_{(h,\bar{h})}(z)\bar{\mathcal{O}}_{(h,\bar{h})}(z')\rangle_{\text{free}} &= \langle(\Phi^h\bar{\Phi}^{\bar{h}})(z)(\bar{\Phi}^h\Phi^{\bar{h}})(z')\rangle_{\text{free}} \\ &= c_{\mathcal{O}}\Bigg(\frac{1}{16}\bar{D}^2D^2 + \frac{\Delta-\omega}{4\Delta}\bar{D}^{\dot{\alpha}}\sigma^\mu_{\alpha\dot{\alpha}}D^\alpha\partial_\mu + \\ &\quad + \frac{(\Delta-\omega)(\Delta-\omega-2)}{4\Delta(\Delta-1)}\Box\Bigg)\frac{\delta^4(\theta-\theta')}{|x-x'|^{2\Delta}}\,,\end{aligned} \tag{3.19}$$

where we have set $\Delta = h + \bar{h}$ for the total dimension and $\omega = h - \bar{h}$ for the chiral weight. $c_{\mathcal{O}}$ is an appropriate tree-level factor.

Because of superconformal invariance the same form is also valid in the interacting theory. Given the assumption[7] that the operators of interest are quasi-primary in the BMN limit the only difference between the free and the interacting cases lies in the scaling

[7] As we mentioned earlier, this assumption appears to be valid only in the infinite J limit. The BMN operators are not quasi-primary at finite J and they should receive $1/J$ corrections.

JHEP06(2003)015

dimensions, which may become anomalous. Similar moduli dependent contributions to the chiral weights do not appear for the following reason. In the presence of $\mathcal{N} = 1$ superconformal invariance the chiral weights of quasi-primary operators are proportional to their $\mathrm{U}(1)_R$ charges [42] and the latter are not expected to receive corrections at any order in perturbation theory. Indeed, since the R-charge of a generic composite operator is the sum of the R-charges of its constituents (see for instance the recent discussion in [43]), R-charge corrections to the BMN operators $\mathcal{U}_J^a$ and $\mathcal{O}_J^a$ in (3.9), (3.10) would imply that the R-charges of the constituent chiral superfields Ω, Ψ^a get renormalized. This type of renormalization cannot occur, however, because the presence of the exact $\mathrm{U}(1)_R$ symmetry fixes the R-charge of the perturbing superpotential (3.4) to be two. The R-symmetry is part of the superconformal algebra and remains exact at any point of the moduli space. In summary, we conclude that the only modification of (3.19) in the interacting theory is the substitution of the canonical dimension Δ by $\Delta + \gamma$. The overall factor $c_{\mathcal{O}}$ becomes in the planar limit a function that generically depends on the couplings $g^2 N, g'^2 N, h_1\sqrt{N}, h_2\sqrt{N}$ and the conformal weights $h, \bar{h}$.

Hence, the full interacting counterpart of eq. (3.19) reads

$$\begin{aligned}\langle \mathcal{O}_{(h,\bar{h})}(z)\bar{\mathcal{O}}_{(h,\bar{h})}(z')\rangle &= c_{\mathcal{O}}(g,g',h_1,h_2;N,h,\bar{h}) \times \\ &\quad \times \left(\frac{1}{16}\bar{D}^2 D^2 + \frac{\Delta+\gamma-\omega}{4(\Delta+\gamma)}\bar{D}^{\dot{\alpha}}\sigma^{\mu}_{\alpha\dot{\alpha}}D^{\alpha}\partial_\mu + \right. \\ &\quad \left. + \frac{(\Delta+\gamma-\omega)(\Delta+\gamma-\omega-2)}{4(\Delta+\gamma)(\Delta+\gamma-1)}\Box\right)\frac{\delta^4(\theta-\theta')}{|x-x'|^{2(\Delta+\gamma)}} \,. \end{aligned} \tag{3.20}$$

We are now ready to apply this general expression on the two-point functions that appear in eq. (3.15).

As a more straightforward situation we would like to begin with the analysis of the special case of zero h_2 coupling. We will return to the more generic situation in a moment. For $h_2 = 0$ eq. (3.15) becomes

$$\frac{1}{16}\langle \bar{D}^2\mathcal{U}_J^1(z) D^2\bar{\mathcal{U}}_J^1(z')\rangle = |A_1|^2 \langle \mathcal{O}_{J+1}^2(z)\bar{\mathcal{O}}_{J+1}^2(z')\rangle \,. \tag{3.21}$$

The operator $\mathcal{U}_J^1$ has canonical dimension $\Delta = J+1$, chiral weight $\omega = J-1$ and some anomalous dimension that we denote by $\gamma_{\mathcal{U}}$. As a result, we write

$$\begin{aligned}\langle \bar{D}^2\mathcal{U}_J^1(z) D^2\bar{\mathcal{U}}_J^1(z')\rangle &= \frac{N^{J+1}}{(4\pi^2)^{J+1}} c(g,g',h_1,h_2;N,J)\bar{D}^2 \times \\ &\quad \times \left(\frac{1}{16}\bar{D}^2 D^2 + \frac{2+\gamma_{\mathcal{U}}}{4(J+1+\gamma_{\mathcal{U}})}\bar{D}^{\dot{\alpha}}\sigma^{\mu}_{\alpha\dot{\alpha}}D^{\alpha}\partial_\mu + \right. \\ &\quad \left. + \frac{(2+\gamma_{\mathcal{U}})\gamma_{\mathcal{U}}}{4(J+1+\gamma_{\mathcal{U}})(J+\gamma_{\mathcal{U}})}\Box\right) D^2 \frac{\delta^4(\theta-\theta')}{|x-x'|^{2(J+1+\gamma_{\mathcal{U}})}} \\ &= \frac{N^{J+1}}{(4\pi^2)^{J+1}} c(g,g',h_1,h_2;N,J)\gamma_{\mathcal{U}}(\gamma_{\mathcal{U}}+2)\bar{D}^2 D^2 \frac{\delta^4(\theta-\theta')}{|x-x'|^{2(J+2+\gamma_{\mathcal{U}})}} \,. \end{aligned} \tag{3.22}$$

To get the last equality we used the well-known identity $\bar{D}^3 = 0$.

Similarly, the operator $\mathcal{O}^2_{J+1}$ has $\Delta = \omega = J+2$ and we denote the corresponding anomalous dimension by $\gamma_{\mathcal{O}}$. In the large J limit the anomalous dimensions of the operators $\mathcal{O}^a_J$ and $\mathcal{O}^a_{J+1}$ can be taken to be the same. Thus,

$$\begin{aligned}\langle \mathcal{O}^2_{J+1}(z)\bar{\mathcal{O}}^2_{J+1}(z')\rangle &= \frac{N^{J+2}}{(4\pi^2)^{J+2}}c_2(g,g',h_1,h_2;N,J)\times\\ &\quad\times\left(\frac{1}{16}\bar{D}^2D^2+\frac{\gamma_{\mathcal{O}}}{4(J+2+\gamma_{\mathcal{O}})}\bar{D}^{\dot\alpha}\sigma^\mu_{\alpha\dot\alpha}D^\alpha\partial_\mu+\right.\\ &\quad\left.+\frac{\gamma_{\mathcal{O}}(\gamma_{\mathcal{O}}-2)}{4(J+2+\gamma_{\mathcal{O}})(J+1+\gamma_{\mathcal{O}})}\Box\right)\frac{\delta^4(\theta-\theta')}{|x-x'|^{2(J+2+\gamma_{\mathcal{O}})}}\\ &\sim\frac{1}{16}\frac{N^{J+2}}{(4\pi^2)^{J+2}}c_2(g,g',h_1,h_2;N,J)\bar{D}^2D^2\frac{\delta^4(\theta-\theta')}{|x-x'|^{2(J+2+\gamma_{\mathcal{O}})}}\,. \end{aligned}\qquad(3.23)$$

The last two terms have been dropped in the final equality because they are subleading in the large J limit.

When $h_2 = 0$ the normalization factors c and c_2 are equal, because the operators $\mathcal{U}^1_J$ and $\mathcal{O}^2_{J+1}$ are part of the same supermultiplet. A similar situation also occurs in the $\mathcal{N}=4$ and $\mathcal{N}=2$ examples of refs. [17, 41]. As a result of eqs. (3.21), (3.22) and (3.23) we obtain the following interesting relations. First, the simple requirement that the same power of $|x-x'|$ appears on both sides of eq. (3.15) implies that the anomalous dimensions $\gamma_{\mathcal{U}}$ and $\gamma_{\mathcal{O}}$ have to be the same. Secondly, if we denote the common value of these dimensions by γ^1 we find that it has to obey the equation

$$(\gamma^1)^2+2\gamma^1=\frac{N}{4\pi^2}|A_1|^2.\qquad(3.24)$$

A_1 is the constant that appears in (3.16) and depends on g', h_1 and J. We can solve this simple quadratic equation for γ^1 and obtain an exact expression for the scaling dimension of the operators $\mathcal{U}^1_J$ and $\mathcal{O}^2_J$.

Before doing that however, let us return to the general situation of the deforming superpotential (3.4), with both h_1 and h_2 non-zero, and explain what happens there. On the r.h.s. of (3.15) we have some extra two-point functions that involve the operator $\mathcal{O}^{11}_J$. As we said above, we make the explicit assumption that this operator is quasi-primary in the infinite J approximation. This allows the use of the general equation (3.19). For the operator $\mathcal{O}^{11}_J$ we have $\Delta=\omega=J+2$. We denote its anomalous scaling dimension by $\gamma_{\mathcal{O}^{11}}$ and, similar to the two-point function of $\mathcal{O}^2_J$, we get

$$\langle\mathcal{O}^{11}_J(z)\bar{\mathcal{O}}^{11}_J(z')\rangle\sim\frac{1}{16}\frac{N^{J+2}}{(4\pi^2)^{J+2}}c_{11}(g,g',h_1,h_2;N,J)\bar{D}^2D^2\frac{\delta^4(\theta-\theta')}{|x-x'|^{2(J+2+\gamma_{\mathcal{O}^{11}})}}\,.\qquad(3.25)$$

Moreover, we will allow for a non-diagonal overlap between the quasi-primary operators

$\mathcal{O}^2_{J+1}$ and $\mathcal{O}^{11}_J$ by setting[8]

$$\langle \mathcal{O}^2_{J+1}(z)\bar{\mathcal{O}}^{11}_J(z')\rangle \sim \frac{1}{16}\frac{N^{J+2}}{(4\pi^2)^{J+2}}c_{12}(g,g',h_1,h_2;N,J)\bar{D}^2D^2\frac{\delta^4(\theta-\theta')}{|x-x'|^{2(J+2+\frac{1}{2}(\gamma_{\mathcal{O}^{11}}+\gamma_{\mathcal{O}}))}}\,, \tag{3.26}$$

where c_{12} is a certain function of the couplings.

In this more generic situation, the operator $\mathcal{U}^1_J$ is part of the same supermultiplet as the linear combination of the operators that appear on the r.h.s. of the equations of motion (3.13). This statement alone, however, is not enough to determine the relation between the normalization factors c, c_2, c_{11} and c_{12}. For the special case $h_2 = 0$ the operator $\mathcal{O}^{11}_J$ was absent and we could deduce the relation $c = c_2$. In addition, the leading order perturbative computation of the next subsection yields $c = c_2 = c_{11}$ also with $h_2 \neq 0$. This indicates that the equation $c = c_2 = c_{11}$ is true at a generic point of the moduli space. The present approach, however, does not provide a proof of this fact and in this subsection we are forced to work with generic normalization factors.

Combining the information of eqs. (3.15), (3.22), (3.23), (3.25), and (3.26) we find that the anomalous dimensions $\gamma_{\mathcal{U}}, \gamma_{\mathcal{O}}$ and $\gamma_{\mathcal{O}^{11}}$ have to be the same and we denote them by γ^1. Again, this result has been derived in the large J limit and we expect $1/J$ corrections to lift this degeneracy. Furthermore, we find the generalization of eq. (3.24)

$$(\gamma^1)^2 + 2\gamma^1 = \frac{N}{4\pi^2}\left(\frac{c_2}{c}|A_1|^2 + 9\frac{c_{11}}{c}|h_2|^2 + 3A_1\bar{h}_2\frac{c_{12}}{c} + 3\bar{A}_1h_2\frac{\bar{c}_{12}}{c}\right). \tag{3.27}$$

In the large 't-Hooft and large J limit the ratios of the c-functions are going to be functions of the couplings g'^2, h_1, h_2 and g^2 multiplied by the appropriate power of N or J. We denote them as

$$\begin{aligned}\mathcal{F}_2(g,g',h_1,h_2;N,J) &= \frac{c_2}{c}\,,\\ \mathcal{F}_{11}(g,g',h_1,h_2;N,J) &= \frac{c_{11}}{c}\,,\\ \mathcal{F}_{12}(g,g',h_1,h_2;N,J) &= \frac{c_{12}}{c}\,.\end{aligned} \tag{3.28}$$

They are expected to take finite values in the large J limit, but, as we said, in generic points of the moduli space it is not possible to determine them exactly using the technology of [17]. Their leading order behavior will be determined in perturbation theory in the next subsection.

With this notation eq. (3.27) has only one reasonable solution and we can write it as

$$\gamma^1 = -1 + \sqrt{1 + \frac{N}{4\pi^2}(\mathcal{F}_2|A_1|^2 + 9\mathcal{F}_{11}|h_2|^2 + 3A_1\bar{h}_2\mathcal{F}_{12} + 3\bar{A}_1h_2\bar{\mathcal{F}}_{12})}\,. \tag{3.29}$$

[8]In perturbation theory these operators have equal scaling dimensions at weak coupling and we have to consider the possibility of mixing. Indeed, in higher orders of perturbation theory this seems possible. We will not discuss this perturbative mixing in this section. Instead, we make use of the large J assumption that the operators in question are quasi-primary and on general grounds proceed with the application of eq. (3.19).

JHEP06(2003)015

Before expressing A_1 more explicitly in the large J approximation, it is probably useful to emphasize the following well-known fact. In the large-N limit it is natural to scale the deformation couplings h_1 and h_2 as $1/\sqrt{N}$. Indeed, as was pointed out, for example in [44], in the large 't Hooft limit it is convenient to normalize the $\mathcal{N}=4$ chiral primary operators $\mathcal{O}_p$ with Dynkin labels $(0,p,0)$ as

$$\mathcal{O}_p = N(g_{\rm YM}^2 N)^{-p/2}\,\mathrm{Tr}\left[\Phi^{(i_1}\cdots\Phi^{i_p)}\right]. \tag{3.30}$$

A similar normalization for the $\mathcal{N}=1$ chiral primary operators that appear in (3.4), gives the extra factor $\frac{1}{g_{\rm YM}^3\sqrt{N}}$ and since we keep $g_{\rm YM}$ small and fixed we simply set $h_i=\lambda_i/\sqrt{N}$ for $i=1,2$. The scaling of the overall coefficient g' in front of the $\mathcal{N}=4$ superpotential is a bit more subtle. In the large J and large 't-Hooft limit it is natural to write $g'=g+h_0$, with $g=\sqrt{2}g_{\rm YM}\ll 1$ being the $\mathcal{N}=4$ value of the coupling and treat h_0 on the same footing as h_1 and h_2. Hence, we also scale h_0 as $1/\sqrt{N}$ and we set $h_0=\lambda_0/\sqrt{N}$. The reasons for this choice will become more apparent in the next subsection, where we discuss the perturbative form of the constraint (3.5). As a consequence of these scalings, g' can be simply substituted by g in any expression, where the dominant g dependence does not cancel exactly.

With these conventions we can write the large-N and J limit of γ^1 as

$$\gamma^1 = -1+\sqrt{1+\alpha_1^2-\alpha_2\frac{g\sqrt{\mathcal{F}_2}\sqrt{N}n}{J}+\frac{\mathcal{F}_2 g^2 N n^2}{J^2}}\,, \tag{3.31}$$

where we have set

$$\begin{aligned}\alpha_1^2 &= \frac{1}{4\pi^2}(4\mathcal{F}_2|\lambda_1|^2+9\mathcal{F}_{11}|\lambda_2|^2+6\mathcal{F}_{12}\lambda_1\bar{\lambda}_2+6\bar{\mathcal{F}}_{12}\bar{\lambda}_1\lambda_2)\,,\\ \alpha_2 &= -\frac{2}{\pi\sqrt{\mathcal{F}_2}}(\mathrm{Im}(\lambda_1)+\frac{3}{4}i(\mathcal{F}_{12}\bar{\lambda}_2-\bar{\mathcal{F}}_{12}\lambda_2))\,.\end{aligned} \tag{3.32}$$

The functions $\mathcal{F}_2,\mathcal{F}_{11},\mathcal{F}_{12}$ can in principle depend only on the couplings $\lambda_0,\lambda_1,\lambda_2$ and the finite ratio g^2N/J^2. Use of the constraint (3.5) should allow a further elimination of the dependence on one of these couplings.

In precisely the same way one may also calculate the anomalous dimension γ^2 of the operators $\mathcal{U}_J^2$ and $\mathcal{O}_J^1$. The result is

$$\gamma^2 = -1+\sqrt{1+\alpha_1^2+\alpha_2\frac{g\sqrt{\mathcal{F}_2}\sqrt{N}n}{J}+\frac{\mathcal{F}_2 g^2 N n^2}{J^2}}\,. \tag{3.33}$$

This dimension is different from γ^1. This is another effect of the breaking of the $\mathcal{N}=4$ superconformal symmetry down to $\mathcal{N}=1$. The corresponding anomalous dimensions at the $\mathcal{N}=4$ point are the same as a consequence of the extended supersymmetry. In particular, it has been shown in [30], that all the relevant BMN operators with two Ψ^a insertions belong to the same long supermultiplet. After the $\mathcal{N}=4$-breaking deformation this property disappears. As we see later, from the string theory perspective this difference is due to the appearance of 3-form fluxes along the (Ψ^1,Ψ^2) plane which break the transverse SO(4) symmetry.

Finally, notice that for generic points of the moduli space the anomalous dimensions (3.31) and (3.33) may become imaginary. Using the perturbative values of $\mathcal{F}_2, \mathcal{F}_{11}$ and $\mathcal{F}_{12}$ we can see that this is not happening around the $\mathcal{N}=4$ point. If it does happen deeper into the moduli space it is not necessarily a bad or pathological feature of the theory, but it should be rather interpreted as a sign that we have to use a different coordinate system on the field theory space in order to get a reasonable description.

3.4 Two-point functions and scaling dimensions in perturbation theory

In this subsection we make an independent computation of the anomalous scaling dimensions (3.31) and (3.33) to leading order in perturbation theory. This will also provide the perturbative values of the ratios (3.28).

We work in superspace formalism and consider only planar diagrams. Our perturbative treatment involves four parameters. Two of them are related to the marginal deformations parameterized by h_1 and h_2. In the previous subsection we re-expressed them as λ_1 and λ_2 and they are finite and tunable couplings in the planar limit. There are two more. One of them we denoted by $g' = g + h_0$ and we set $h_0 = \lambda_0/\sqrt{N}$ and the other parameter is proportional to the Yang-Mills coupling. By now, it has been firmly established [24, 26] that what governs the strong t'-Hooft, large R-charge perturbation theory with respect to the $\mathcal{N}=4$ superpotential is not t' Hooft's coupling per se, which becomes infinite, but rather the finite and tunable parameter $\lambda' = g^2_{\rm YM}\frac{N}{J^2}$.

We begin by computing the leading order correction to the two-point function $\langle \mathcal{U}^1_J(z)\, \bar{\mathcal{U}}^1_J(z')\rangle$ in the presence of the exactly marginal deforming superpotential (3.4)

$$W_{\rm def} = h_1 {\rm Tr}(\Psi^1\Psi^2\Omega + \Psi^2\Psi^1\Omega) + h_2 {\rm Tr}\left((\Psi^1)^3 + (\Psi^2)^3 + \Omega^3\right). \tag{3.34}$$

The interacting part of the full action, which involves only the Higgs superfields Ψ^1, Ψ^2, Ω can be written as

$$\int d^4x \int d^2\theta \Big[g' {\rm Tr}((\Psi^1\Psi^2 - \Psi^2\Psi^1)\Omega) + h_1 {\rm Tr}((\Psi^1\Psi^2 + \Psi^2\Psi^1)\Omega) + \\ + h_2 {\rm Tr}((\Psi^1)^3 + (\Psi^2)^3 + \Omega^3)\Big] + {\rm c.c.} \tag{3.35}$$

and the leading non-zero corrections come from the second order terms[9]

$$\int d^4x_1 \int d^2\theta_1 \int d^4x_2 \int d^2\bar{\theta}_2 \Big[(h_1 + g')(\bar{h}_1 - \bar{g}'){\rm Tr}(\Psi^1\Psi^2\Omega(z_1))\, {\rm Tr}(\bar{\Psi}^1\bar{\Psi}^2\bar{\Omega}(z_2)) + \\ + |h_1 + g'|^2 {\rm Tr}(\Psi^1\Psi^2\Omega(z_1))\, {\rm Tr}(\bar{\Psi}^2\bar{\Psi}^1\bar{\Omega}(z_2)) + \\ + |h_1 - g'|^2 {\rm Tr}(\Psi^2\Psi^1\Omega(z_1))\, {\rm Tr}(\bar{\Psi}^1\bar{\Psi}^2\bar{\Omega}(z_2)) + \\ + (h_1 - g')(\bar{h}_1 + \bar{g}'){\rm Tr}(\Psi^2\Psi^1\Omega(z_1))\, {\rm Tr}(\bar{\Psi}^2\bar{\Psi}^1\bar{\Omega}(z_2))\Big] \tag{3.36}$$

and

$$|h_2|^2 \int d^4x_1 \int d^2\theta_1 \int d^4x_2 \int d^2\bar{\theta}_2 \Big[{\rm Tr}(\Psi^1)^3(z_1){\rm Tr}(\bar{\Psi}^1)^3(z_2) + {\rm Tr}\Omega^3(z_1){\rm Tr}\bar{\Omega}^3(z_2)\Big)\Big]. \tag{3.37}$$

[9] We use the convention that the superspace coordinate $z = (x, \theta, \bar{\theta})$ is appropriately truncated to its chiral part (x, θ) when it appears as an argument of a chiral superfield and similarly for antichiral superfields.

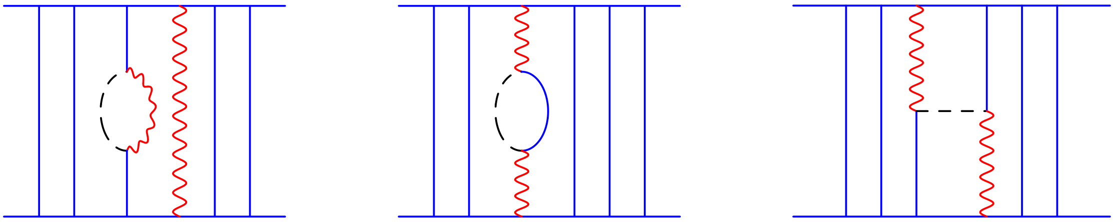

Figure 1: The three types of super-Feynman diagrams that contribute to the leading order anomalous dimensions of $\mathcal{U}_J^1$. Straight lines depict propagators of Ω, wiggly lines propagators of Ψ^1 and syncopated lines propagators of Ψ^2.

Note that there is no mixing between the h_1 and h_2 deformations at this order in perturbation theory. The contribution to the anomalous dimensions from diagrams involing the gauge field multiplet will be taken into account at the end of the computation.

The general form (3.9) of the operators $\mathcal{U}_J^1$ implies that we need to compute amplitudes of the form $\langle \Omega^l \bar{\Psi}^1 \Omega^{J-l}(z)\ \bar{\Omega}^m \Psi^1 \bar{\Omega}^{J-m}(z') \rangle$ with insertions of the interacting terms (3.36) and (3.37). In order to find the full two-point function $\langle \mathcal{U}_J^1(z)\ \bar{\mathcal{U}}_J^1(z') \rangle$ we have to incorporate the phases $e^{i(l-m)\varphi}$ and sum over the integers l and m that provide planar diagrams. In order to make the computation more transparent we first ignore the fact that the fields are matrices in the SU(N) Lie algebra and reinstate the relevant group-theoretical factors later.

After using Wick's theorem and the chiral superfield propagators (3.18), we obtain three types of diagrams from (3.36). We can write them as

$$\begin{aligned}
&\frac{1}{16^{J+4}} \frac{1}{(4\pi^2)^{J+4}} \int d^4x_1 \int d^2\theta_1 \int d^4x_2 \int d^2\bar{\theta}_2 \left(\bar{D}^2 D^2 F(z,z')\right)^{J-1} \left(D^2 \bar{D}^2 F(z,z')\right) \times \\
&\times \left(D^2 \bar{D}^2 F(z_2,z)\right) \left(\bar{D}^2 D^2 F(z_1,z')\right) \left(\bar{D}^2 D^2 F(z_1,z_2)\right)^2 \\
&\frac{1}{16^{J+4}} \frac{1}{(4\pi^2)^{J+4}} \int d^4x_1 \int d^2\theta_1 \int d^4x_2 \int d^2\bar{\theta}_2 \left(\bar{D}^2 D^2 F(z,z')\right)^{J} \left(\bar{D}^2 D^2 F(z_1,z)\right) \times \\
&\times \left(D^2 \bar{D}^2 F(z_2,z')\right) \left(\bar{D}^2 D^2 F(z_1,z_2)\right)^2 \\
&\frac{1}{16^{J+4}} \frac{1}{(4\pi^2)^{J+4}} \int d^4x_1 \int d^2\theta_1 \int d^4x_2 \int d^2\bar{\theta}_2 \left(\bar{D}^2 D^2 F(z,z')\right)^{J-1} \left(D^2 \bar{D}^2 F(z_2,z)\right) \times \\
&\times \left(\bar{D}^2 D^2 F(z_1,z')\right) \left(\bar{D}^2 D^2 F(z_1,z_2)\right) \left(D^2 \bar{D}^2 F(z_2,z')\right) \left(\bar{D}^2 D^2 F(z_1,z)\right). \qquad (3.38)
\end{aligned}$$

For convenience we use the notation

$$F(z,z') = \frac{\delta^4(\theta-\theta')}{|x-x'|^2} \qquad (3.39)$$

and by convention the superspace derivatives $\bar{D}$ and D are taken to act on the first argument of F. The associated super-Feynman diagrams are shown in figure 1. The first two diagrams are wave-function renormalizations of the fields Ω and Ψ^1 respectively and they are phase-independent in the planar limit. The third diagram interchanges the position of $\bar{\Psi}^1$ and is responsible for the level n dependence of the anomalous dimension.

JHEP06(2003)015

In order to compute the anomalous dimension of an operator to leading order we are instructed to compute the relevant two-point function and, if working in dimensional regularization, extract the $1/\epsilon$ divergence. In a different regularization scheme, e.g. with a UV cutoff, this divergence corresponds to a logarithmic correction of the propagator. After the appropriate renormalization, the final result takes the generic form (3.20).

A well-known subtlety in the above procedure has to do with operators that share the same canonical dimensions. Usually such operators mix on the quantum level and the corresponding two-point functions are no longer diagonal. In that case, the correct anomalous dimensions result from the diagonalization of the mixing matrix. This appears to be a problem in our case because it seems that in general we have to diagonalize an infinite dimensional mixing matrix. The same problem is also encountered at the $\mathcal{N}=4$ point. Our attitude towards this is the following. As in section 3.3 we make the large J assumption that the operators in question are always quasi-primary and this allows the computation of the anomalous dimensions from a simple two-point function calculation.

In our superspace formalism computation it is convenient to focus on the theta-independent piece of the superfield two-point function. This corresponds to the two-point function between the bottom components of each operator and by supersymmetry all the fields in the full multiplet should have the same anomalous dimensions. In general, if the 1-loop corrected propagator between the bottom components of a superfield operator $\mathcal{O}_{(h,\bar{h})}$ with canonical dimension $\Delta = h+\bar{h}$ has the form

$$\langle \mathcal{O}_{(h,\bar{h})}|_{\theta=\bar{\theta}=0}(z)\ \bar{\mathcal{O}}_{(h,\bar{h})}|_{\theta'=\bar{\theta}'=0}(z')\rangle = c_{\mathcal{O}}^{(1)}\frac{1}{|x-x'|^{2\Delta}}\left(1+\gamma_{(1)}\frac{1}{\epsilon}\right), \tag{3.40}$$

the leading contribution to its anomalous dimension is given by the coefficient $\gamma_{(1)}$ of the UV divergent term. The divergence is regularized by the standard dimensional regularization continuation to $d=4-2\epsilon$ dimensions.

We can now proceed with the computation of the first diagram. The free piece is given by the general form (3.19) after setting $\Delta = J$ and $\omega = J-2$ and in the planar limit the associated factor $c_{\mathcal{O}}$ is $(\frac{N}{4\pi^2})^J$. The computation of the superspace integral involves the free chiral superfield propagators [40]

$$\begin{aligned}
\bar{D}^2 D^2 F(z_1,z_2) &= 16 e^{i(\theta_1\sigma^n\bar{\theta}_1+\theta_2\sigma^n\bar{\theta}_2-2\theta_1\sigma^n\bar{\theta}_2)\partial_n}\frac{1}{|x_1-x_2|^2}\\
D^2\bar{D}^2 F(z_1,z_2) &= 16 e^{-i(\theta_1\sigma^n\bar{\theta}_1+\theta_2\sigma^n\bar{\theta}_2-2\theta_2\sigma^n\bar{\theta}_1)\partial_n}\frac{1}{|x_1-x_2|^2}.
\end{aligned} \tag{3.41}$$

Remember that we always take the superspace derivatives to act on the first argument of $F(z_1,z_2)$.

Then we can easily perform the fermionic integrations to obtain

$$\begin{aligned}
&\int d^2\theta_1 \int d^2\bar{\theta}_2 (D^2\bar{D}^2F(z_2,z))(\bar{D}^2D^2F(z_1,z'))(\bar{D}^2D^2F(z_1,z_2))^2 = \\
&16^4\left[\Box\Delta^2_{x_1x_2}\left(\Delta_{xx_2} - i\theta\sigma^n\bar{\theta}\partial^{x_2}_n\Delta_{xx_2}+\frac{1}{4}\theta^2\bar{\theta}^2\Box\Delta_{xx_2}\right)\times\right.\\
&\left.\times\left(\Delta_{x'x_1} - i\theta'\sigma^m\bar{\theta}'\partial^{x_1}_m\Delta_{x'x_1}+\frac{1}{4}\theta'^2\bar{\theta}'^2\Box\Delta_{x'x_1}\right)\right] +
\end{aligned}$$

$$
\begin{aligned}
&+\left[\theta^2\bar\theta^2\Delta^2_{x_1x_2}\Box\Delta_{xx_2}\Box\Delta_{x'x_1}\right]+\\
&+[4i\Delta_{x_1x_2}(i\theta\partial_n^{x_2}\Delta_{xx_2}+(\theta\sigma^l\bar\theta)\theta\partial_l^{x_2}\partial_n^{x_2}\Delta_{xx_2})\sigma^n\partial_m^{x_2}\Delta_{x_1x_2}\bar\sigma^m\sigma^k\times\\
&\quad\times(i\bar\theta'\partial_k^{x_1}\Delta_{x'x_1}+(\theta'\sigma^s\bar\theta')\bar\theta'\partial_s^{x_1}\partial_k^{x_1})],
\end{aligned}
\tag{3.42}
$$

where

$$
\Delta_{x_1x_2}=\frac{1}{|x_1-x_2|^2}\,. \tag{3.43}
$$

The spacetime integrations of the theta-independent piece give

$$
\int d^4x_1\int d^4x_2\,16^4\Delta_{xx_2}\Delta_{x'x_1}\Box\Delta^2_{x_1x_2}=16^4\frac{8\pi^4}{\epsilon}\frac{1}{|x-x'|^2} \tag{3.44}
$$

where we have used dimensional regularization and the general formula (see, for example, [45])

$$
\int d^dx\frac{1}{|x|^{2a}|x-y|^{2b}}=\pi^2\frac{\Gamma(a+b-\frac{d}{2})}{\Gamma(a)\Gamma(b)}\frac{\Gamma(\frac{d}{2}-a)\Gamma(\frac{d}{2}-b)}{\Gamma(d-a-b)}\frac{1}{|y|^{2(a+b-\frac{d}{2})}}\,, \tag{3.45}
$$

in $d=4-2\epsilon$ dimensions.

The second diagram can be performed in a similar way. Basically, the integrand differs from the one we just computed by the interchange $z\leftrightarrow z'$ and the theta-independent piece gives again (3.44) . The contribution of the free contractions is of the type (3.19) with $\Delta=\omega=J$ and the relevant prefactor in the planar limit is $(\frac{N}{4\pi^2})^J$. It is also important to notice that for both the first and second diagram there is only one planar contraction, which comes from the second and third term in (3.36). In the two-point function $\langle\Omega^l\bar\Psi^1\Omega^{J-l}(z)\;\bar\Omega^m\Psi^1\bar\Omega^{J-m}(z')\rangle$ this requires $l=m$. As a result, such diagrams do not depend on the level n and the summation over l and m just gives a factor of J which is common to all diagrams and which along with the same factor of the tree level correlator can be absorbed in the overall normalization of the operator [16] .

Because all the fields under consideration are in the adjoint representation of the SU(N) gauge group, we also have to take into account the relevant group theory factors. For a field $\Phi_{ab}=\Phi_AT^A_{ab}$, with $A=1,\ldots,N^2-1$ and T^A_{ab} a set of $N\times N$ traceless and hermitian matrices spanning the Lie algebra of SU(N), the free propagator comes with a factor

$$
\langle\Phi_{ab}\bar\Phi_{a'b'}\rangle\sim\left(\delta_{ab'}\delta_{a'b}-\frac{1}{N}\delta_{ab}\delta_{a'b'}\right). \tag{3.46}
$$

The second term at the right is subleading in large-N computations and can be neglected. Due to summation over indices in the loop, contractions involving the interaction vertices come with an additional factor of N compared to the free propagator, and after incorporating them into the full amplitude they result to an extra factor of N^2.

Putting everything together, we conclude that the UV divergent term in the two-point function of the bottom components coming from the first and the second diagram is

$$
\begin{aligned}
&(1+1)\;2(|h_1|^2+|g'|^2)\left(\frac{N}{4\pi^2}\right)^J\frac{1}{(4\pi^2)^4}N^2\frac{8\pi^4}{\epsilon}\times\\
&\times\frac{1}{|x-x'|^{2(J+1)}}=\left(\frac{N}{4\pi^2}\right)^{J+1}\frac{1}{\epsilon}\left[\frac{N}{2\pi^2}\left(|h_1|^2+|g'|^2\right)\right]\frac{1}{|x-x'|^{2(J+1)}}\,.
\end{aligned}
\tag{3.47}
$$

The contribution of the third diagram can be deduced in a similar way. The free contractions in this case take the form (3.19) with $\Delta = \omega = J-1$ and there is a usual prefactor $(\frac{N}{4\pi^2})^{J-1}$. The loop integral is now more complicated and we write down explicitly only the theta-independent part, which is sufficient for our purposes. This reads

$$\int d^4x_1 \int d^4x_2 \; 16^5 \Delta_{xx_1}\Delta_{x'x_1}\Delta_{xx_2}\Delta_{x'x_2}\Box\Delta_{x_1x_2} = 16^5 \frac{8\pi^4}{\epsilon}\frac{1}{|x-x'|^4}\,. \tag{3.48}$$

We used the identity $\Box\Delta_{x_1x_2} = -4\pi^2\delta(x_1-x_2)$ and the general formula (3.45) .

Since we work in the planar limit, only amplitudes with $l-m=1$ from the first vertex and $l-m=-1$ from the fourth vertex of (3.36) should be kept. Consequently, this diagram is phase-dependent and the relevant factor is $e^{i\varphi}(h_1+g')(\bar{h}_1-\bar{g'}) + e^{-i\varphi}(h_1-g')(\bar{h}_1+\bar{g'}) = 2\cos\varphi\,(|h_1|^2-|g'|^2) + 4\sin\varphi\mathrm{Im}(\bar{g'}h_1)$. The total group theoretical factor is N^3.

Assembling every piece from the third diagram we get the following contribution to the bottom component UV divergence

$$\begin{aligned}&\big(2\cos\varphi(|h_1|^2-|g'|^2) + 4\sin\varphi\mathrm{Im}(\bar{g'}h_1)\big)\left(\frac{N}{4\pi^2}\right)^{J-1}\frac{1}{(4\pi^2)^5}\,N^3\frac{8\pi^4}{\epsilon}\frac{1}{|x-x'|^{2(J+1)}} = \\ &= \left(\frac{N}{4\pi^2}\right)^{J+1}\frac{1}{\epsilon}\left[\frac{N}{4\pi^2}(\cos\varphi\;(|h_1|^2-|g'|^2) + 2\sin\varphi\mathrm{Im}(\bar{g'}h_1))\right]\frac{1}{|x-x'|^{2(J+1)}}\,.\end{aligned} \tag{3.49}$$

As a result, the total correction due to (3.36) can be written as

$$\left(\frac{N}{4\pi^2}\right)^{J+1}\frac{1}{\epsilon}\Big[\frac{N}{4\pi^2}\left((2+\cos\varphi)|h_1|^2 + (2-\cos\varphi)|g'|^2 + 2\sin\varphi\mathrm{Im}(\bar{g'}h_1)\right)\Big]\frac{1}{|x-x'|^{2(J+1)}}\,. \tag{3.50}$$

A similar analysis can be performed for the diagrams that come from (3.37). From the first term we obtain only one connected diagram, which is identical to the second diagram from (3.36) and an extra symmetry factor of 9. In the planar limit only amplitudes with $l=m$ should be kept. From the second term we get two diagrams; one is exactly the same as the first diagram of (3.36) and the other is given by

$$\begin{aligned}&\frac{1}{16^{J+4}}\frac{1}{(4\pi^2)^{J+4}}\int d^4x_1\int d^2\theta_1\int d^4x_2\int d^2\bar{\theta}_2 (\bar{D}^2D^2F(z,z'))^{J-2}(D^2\bar{D}^2F(z,z'))\times \\ &\times(\bar{D}^2D^2F(z_1,z'))^2\Big(D^2\bar{D}^2F(z_2,z')\Big)^2(\bar{D}^2D^2F(z_1,z_2))\,.\end{aligned} \tag{3.51}$$

Again, for both of them there is an extra symmetry factor of 9.

Computing the theta-independent term we get

$$16^5\int d^4x_1\int d^4x_2\;\Delta^2_{xx_2}\Delta^2_{x'x_1}\Box\Delta_{x_1x_2} = 16^5\frac{8\pi^4}{\epsilon}\frac{1}{|x-x'|^4}\,. \tag{3.52}$$

After combining everything, we conclude that the divergent piece in the bottom component of the two-point function due to (3.37) is

$$\begin{aligned}&9|h_2|^2(1+1+1)\left(\frac{N}{4\pi^2}\right)^J\frac{1}{(4\pi^2)^4}\;N^2\frac{8\pi^4}{\epsilon}\frac{1}{|x-x'|^{2(J+1)}} = \\ &\left(\frac{N}{4\pi^2}\right)^{J+1}\frac{1}{\epsilon}\left[\frac{N}{4\pi^2}\frac{27}{2}|h_2|^2\right]\frac{1}{|x-x'|^{2(J+1)}}\,.\end{aligned} \tag{3.53}$$

JHEP06(2003)015

Adding (3.50) with (3.53) and comparing with the general expression (3.40) gives the value of the anomalous dimension to first order

$$\gamma^1_{(1)} = \frac{N}{4\pi^2}\left[\Big((2+\cos\varphi)\;|h_1|^2 + (2-\cos\varphi)\;|g'|^2) + 2\sin\varphi\;\mathrm{Im}(\bar{g'}h_1)\Big) + \left(\frac{27}{2}|h_2|^2\right)\right]. \tag{3.54}$$

In order to compare this expression to the exact result (3.31) of subsection 3.2 we have to take into account the contribution of diagrams involving the gauge field multiplet, which have been ignored so far, and we also have to use the leading order form of the constraint (3.5) to eliminate the λ_0 dependence and express the result only in terms of λ_1, λ_2 and g. The vector multiplet contribution is, of course, n-independent and as in [16] it can be simply deduced from the requirement that at the $\mathcal{N}=4$ point the operators without phases are protected and hence they have vanishing anomalous dimensions. This implies that we have to make a shift proportional to the gauge coupling g^2 in the previous expression, so that the final result reads

$$\gamma^1_{(1)} = \frac{N}{4\pi^2}\left[\Big((2+\cos\varphi)\;|h_1|^2 + (2-\cos\varphi)\;|g'|^2 - g^2 + 2\sin\varphi\;\mathrm{Im}(\bar{g'}h_1)\Big) + \left(\frac{27}{2}|h_2|^2\right)\right]. \tag{3.55}$$

The leading order form of the constraint equation (3.5) can be determined by computing the perturbative anomalous dimension of the operator Π_J (3.6) and require that it vanishes. We now proceed to determine this anomalous dimension.

Perturbative corrections to the two-point function $\langle\Omega^J(z)\;\bar{\Omega}^J(z')\rangle$ are once again due to (3.36) and (3.37) . From (3.36) we obtain one diagram that corresponds to a wave-function renormalization of Ω. This is similar to the first diagram we encountered in the computation of $\langle\mathcal{U}^1_J(z)\;\bar{\mathcal{U}}^1_J(z')\rangle$ and can be written as

$$\frac{1}{16^{J+2}}\frac{1}{(4\pi^2)^{J+2}}\int d^4x_1\int d^2\theta_1\int d^4x_2\int d^2\bar{\theta}_2(\bar{D}^2D^2F(z,z'))^{J-1}(D^2\bar{D}^2F(z_2,z))\times$$
$$\times(\bar{D}^2D^2F(z_1,z'))(\bar{D}^2D^2F(z_1,z_2))^2\,. \tag{3.56}$$

The corresponding theta-independent piece has the UV divergent term

$$2(|h_1|^2+|g'|^2)\left(\frac{N}{4\pi^2}\right)^{J-1}\frac{1}{(4\pi^2)^4}\,N^2\frac{8\pi^4}{\epsilon}\,\frac{1}{|x-x'|^{2J}} =$$
$$\left(\frac{N}{4\pi^2}\right)^J\frac{1}{\epsilon}\left[\frac{N}{4\pi^2}\;(|h_1|^2+|g'|^2)\right]\frac{1}{|x-x'|^{2J}}\,. \tag{3.57}$$

From the second term of (3.37) we obtain two diagrams. These are similar to the corresponding diagrams in the previous computation of $\langle\mathcal{U}^1_J(z)\;\bar{\mathcal{U}}^1_J(z')\rangle$, except that the former have an extra propagator $(D^2\bar{D}^2F(z,z'))$ due to the Ψ^1 insertions. Comparison with (3.53) gives the anomalous dimension contribution

$$9|h_2|^2(1+1)\left(\frac{N}{4\pi^2}\right)^J\frac{1}{(4\pi^2)^4}N^2\frac{8\pi^4}{\epsilon}\frac{1}{|x-x'|^{2(J+1)}} =$$
$$\left(\frac{N}{4\pi^2}\right)^{J+1}\frac{1}{\epsilon}\left[\frac{N}{4\pi^2}\frac{18}{2}|h_2|^2\right]\frac{1}{|x-x'|^{2(J+1)}}\,. \tag{3.58}$$

Consequently, the leading order correction to the anomalous dimension of Π_J is

$$\gamma_{\Pi_J(1)} = \frac{N}{4\pi^2}(|g'|^2 - g^2 + |h_1|^2 + 9|h_2|^2)\,. \tag{3.59}$$

As before, we have subtracted an appropriate constant term in order to account for the contribution of diagrams involving the gauge multiplet. As we said, superconformal invariance requires the vanishing of this anomalous dimension. The resulting constraint is the leading order form of the constraint equation (3.5)

$$g'^2 N - g^2 N + |\lambda_1|^2 + 9|\lambda_2|^2 = 0\,. \tag{3.60}$$

This relation can also be found in [35, 36]. We can use it to eliminate the h_0 dependence from the anomalous dimension (3.55) and thus we get

$$\gamma^1_{(1)} = \frac{N}{4\pi^2}\left[2|\lambda_1|^2 + \frac{9}{2}|\lambda_2|^2 + \frac{1}{\pi}\mathrm{Im}(\lambda_1)\frac{g\sqrt{N}n}{J} + \frac{1}{2}\frac{g^2Nn^2}{J^2}\right]. \tag{3.61}$$

This result agrees with the outcome of the exact computation (3.31) of the previous subsection and also provides the leading order values of the ratios (3.28):

$$\mathcal{F}_2 = \mathcal{F}_{11} = 1 \quad \text{and} \quad \mathcal{F}_{12} = 0\,. \tag{3.62}$$

4. String theory backgrounds from gauge theory anomalous dimensions

The purpose of this section is the reconstruction of a string theory from the gauge theory data that were obtained above. The working assumption of our analysis is that the correspondence between gauge theory operators and string states that was proposed in [16] and outlined in our case in section 2 remains valid even when one deforms away from the $\mathcal{N}=4$ point. We have traced the effect of the deformation on the gauge theory side and now we would like to trace this effect also on the string theory side. In the large J limit this is possible because of the nature of the gauge theory/string theory correspondence, which essentially amounts to the identification of the scaling weights of certain near-BPS operators on the gauge theory side with the spectrum of a light-cone worldsheet theory. This allows the reverse-engineering of a "dual" string background directly from gauge theory data. The word dual is inside quotation marks, because we will soon see that this reverse-engineering process does not produce a unique background in the infinite J limit.

4.1 Bosonic sector

The bosonic part of the light-cone worldsheet theory at the $\mathcal{N}=4$ point is given by the action

$$\mathcal{S}_0 = \mathcal{S}_{\vec{r},0} + \mathcal{S}_{z^1,0} + \mathcal{S}_{z^2,0}\,, \tag{4.1}$$

where there are three distinct parts. $\mathcal{S}_{\vec{r},0}$ is related to the four directions that descend from the AdS_5 part of the geometry after we take the Penrose limit and reads

$$\mathcal{S}_{\vec{r},0} = \frac{1}{2\pi\alpha'}\int d\tau \int_0^{2\pi\alpha' p^+} d\sigma \frac{1}{2}\left(\dot{\vec{r}}^2 - \vec{r}'^2 - \vec{r}^2\right), \tag{4.2}$$

gauge theory operator	$\Delta - J$
$\sum_l e^{il\varphi}\Omega\cdots\Omega\Psi^1\Omega\cdots\Omega$	$-1+\sqrt{1+\alpha_1^2+\alpha_2\frac{g\sqrt{\mathcal{F}_2}\sqrt{N}n}{J}+\frac{\mathcal{F}_2 g^2 N n^2}{J^2}}$
$\sum_l e^{il\varphi}\Omega\cdots\Omega\bar{\Psi}^1\Omega\cdots\Omega$	$1+\sqrt{1+\alpha_1^2-\alpha_2\frac{g\sqrt{\mathcal{F}_2}\sqrt{N}n}{J}+\frac{\mathcal{F}_2 g^2 N n^2}{J^2}}$
$\sum_l e^{il\varphi}\Omega\cdots\Omega\Psi^2\Omega\cdots\Omega$	$-1+\sqrt{1+\alpha_1^2-\alpha_2\frac{g\sqrt{\mathcal{F}_2}\sqrt{N}n}{J}+\frac{\mathcal{F}_2 g^2 N n^2}{J^2}}$
$\sum_l e^{il\varphi}\Omega\cdots\Omega\bar{\Psi}^2\Omega\cdots\Omega$	$1+\sqrt{1+\alpha_1^2+\alpha_2\frac{g\sqrt{\mathcal{F}_2}\sqrt{N}n}{J}+\frac{\mathcal{F}_2 g^2 N n^2}{J^2}}$

Table 1: Anomalous dimensions associated to φ-dependent insertions of $\Psi^1, \bar{\Psi}^1, \Psi^2$ and $\bar{\Psi}^2$ in Π_J. The fields appearing in the above operators are the lowest bosonic components of the corresponding superfields of the previous section.

with $\alpha' p^+ = \frac{J}{g\sqrt{N}}$. This part has the standard spectrum of four massive decoupled oscillators. On the other hand, $\mathcal{S}_{z^a,0}$ (for $a = 1, 2$) are each related to two of the four transverse coordinates that descend from the S^5 part of the full geometry. We have

$$\mathcal{S}_{z^a,0} = \frac{1}{2\pi\alpha'}\int d\tau \int_0^{2\pi\alpha' p^+} d\sigma \frac{1}{2}\left(\dot{z}^a\dot{\bar{z}}^a - z'^a\bar{z}'^a + i\left(\bar{z}^a\dot{z}^a - z^a\dot{\bar{z}}^a\right)\right). \tag{4.3}$$

These parts give the Landau spectra that were discussed in section 2.

From the spectra that were derived in section 3, it is immediately clear how the above worldsheet actions should be deformed to reproduce them. Since the operators with insertions of $D_i\Omega$ and no phases are still protected in the deformed theory, being descendants of Π_J, they continue to have $\Delta - J = 1$. Hence, the corresponding part of the worldsheet action involving the $\vec{r}$-part still consists of four decoupled oscillators and it can be written as

$$\mathcal{S}_{\vec{r},h_1h_2} = \frac{1}{2\pi\alpha'}\int d\tau \int_0^{2\pi\alpha' p^+} d\sigma \frac{1}{2}(\dot{\vec{r}}^2 - \vec{r}'^2 - \vec{r}^2). \tag{4.4}$$

Nevertheless, consistency with the results of table 1, requires the modified lightcone momentum $p^+ = \frac{J}{\alpha' g\sqrt{\mathcal{F}_2}\sqrt{N}}$.

For the z^a-part we should reproduce the spectra associated to the anomalous dimensions of eqs. (3.31), (3.33). These spectra together with the corresponding gauge theory operators are summarized in table 1. It is straightforward to verify that they can be reproduced by the following worldsheet action

$$\sum_{a=1}^{2}\mathcal{S}_{z^a,h_1,h_2} = \frac{1}{2\pi\alpha'}\sum_{a=1}^{2}\int d\tau \int_0^{2\pi\alpha' p^+} d\sigma \frac{1}{2}(\dot{z}^a\dot{\bar{z}}^a - z'^a\bar{z}'^a + \\ + i(\bar{z}^a\dot{z}^a - z^a\dot{\bar{z}}^a) + \alpha_1^2 z^a\bar{z}^a + \frac{1}{2}(-1)^{a+1}\alpha_2 i(z^a\bar{z}'^a - \bar{z}^a z'^a)). \tag{4.5}$$

Because of the way the minus sign of the $\frac{g\sqrt{\mathcal{F}_2}\sqrt{N}n}{J}$ term appears in the anomalous dimensions, there is a subtle difference in the correspondence between string states and gauge theory operators as it appears here and in eqs. (2.32) and (2.35) at the $\mathcal{N} = 4$ point. If we name c^a and $\bar{c}^a$ the oscillators of the worldsheet fields z^a and $\bar{z}^a$ that appear in (4.5) then with respect to the oscillators b^a and $\bar{b}^a$ that appear in (2.32) and (2.35) we have the twisted relation

$$\begin{aligned} b^1 &= c^2 , \qquad b^2 = c^1 , \\ \bar{b}^1 &= \bar{c}^1 , \qquad \bar{b}^2 = \bar{c}^2 . \end{aligned} \tag{4.6}$$

The bosonic worldsheet action of this section shows that the deformation has turned on a 2-form NS-NS B-field with constant field strength along the (z^1, z^2) plane and has modified the metric accordingly. The validity of the supergravity equations of motion in addition requires the modification of the 5-form field strength and/or the presence of a 3-form R-R flux. For the full determination of these components of the dual background we also need to analyze the fermionic string sector, which we now proceed to do explicitly. In this analysis it is convenient to ignore the supersymmetric partners of the "magnetic" terms of the bosonic action (4.5). This means that implicitly we choose to work on the rotated coordinate system of eq. (2.23) and the spectra we would like to reproduce in string theory do not involve the ± 1 twist outside the square root.

4.2 Fermionic sector

In this subsection we provide an analysis of the fermionic string sector. Our conventions follow closely those of [46] and [23]. In the light-cone gauge, the fermionic part of the worldsheet action in the presence of 5-form R-R and 3-form NS-NS and R-R fluxes becomes

$$\begin{aligned} S_F = -\frac{i}{\pi}\alpha' p^+ \int d\tau \int_0^{2\pi\alpha' p^+} \bigg(& \bar{\theta}\Gamma_-(\partial_\tau\theta + \rho\partial_\sigma\theta) + \frac{1}{8}\bar{\theta}\Gamma_-\, \not{H}_3\rho\theta + \\ & + \bar{\theta}\Gamma_-\, \not{F}_3\rho_1\theta + \frac{1}{240}\bar{\theta}\Gamma_-\, \not{F}_5\rho_0\theta \bigg) . \end{aligned} \tag{4.7}$$

θ^I (for $I = 1, 2$) denote two 16-component Majorana-Weyl spinors. ρ, ρ_0 and ρ_1 are two-dimensional gamma matrices, which can be expressed in terms of the Pauli matrices σ_i as

$$\rho_0 = i\sigma_2 , \qquad \rho_1 = \sigma_1 , \qquad \rho = \rho_0\rho_1 = \sigma_3 . \tag{4.8}$$

We have also defined

$$\not{H}_3 = H_{+ij}\Gamma_{ij} , \qquad \not{F}_3 = F_{+ij}\Gamma_{ij} , \qquad \not{F}_5 = F_{+ijkl}\Gamma_{ijkl} . \tag{4.9}$$

With latin characters $i, j, \cdots$ we symbolize the transverse coordinates $\vec{r}$ $(i = 1, 2, 3, 4)$, z^a $(i = 5, 7)$ and $\bar{z}^a$ $(i = 6, 8)$.

From the form of the light-cone energies listed in table 1, we anticipate NS-NS and R-R 3-form fluxes with non-zero components only along $+56$ and $+78$ and hence we set

$$H_{+56} = H_{56} , \qquad H_{+78} = H_{78} , \qquad F_{+56} = F_{56} , \qquad F_{+78} = F_{78} \tag{4.10}$$

and

$$F_{+1234} = F_{+5678} = M . \tag{4.11}$$

Accordingly, the fermionic action (4.7) becomes

$$S_F = -\frac{i}{\pi}\alpha' p^+ \int d\tau \int_0^{2\pi\alpha' p^+} \times$$
$$\times\Big(\theta^1\Gamma_-\partial_+\theta^1 + \theta^2\Gamma_-\partial_-\theta^2 + +\frac{1}{2}\theta^1\Gamma_-(F_{56}\Gamma_{56} + F_{78}\Gamma_{78})\theta^2 + \tag{4.12}$$
$$\frac{1}{4}\theta^1\Gamma_-(H_{56}\Gamma_{56} + H_{78}\Gamma_{78})\theta^1 - -\frac{1}{4}\theta^2\Gamma_-(H_{56}\Gamma_{56} + H_{78}\Gamma_{78})\theta^2 - 2M\theta^1\Gamma_-\Gamma_{5678}\theta^2\Big)$$

and $\partial_\pm = \partial_\tau \pm \partial_\sigma$. By Fourier expanding the fermionic coordinates

$$\theta^I(\tau,\sigma) = \sum_n \theta_n^I(\tau) e^{i\frac{n}{\alpha' p^+}\sigma} \tag{4.13}$$

and substituting into the equations of motion that derive from (4.12) we get

$$\begin{aligned}
\dot\theta_n^1 + \frac{1}{4}\Big(F_{56}\Gamma_{56} + F_{78}\Gamma_{78} - 4M\Gamma_{5678}\Big)\theta_n^2 + \frac{1}{4}\Big(H_{56}\Gamma_{56} + H_{78}\Gamma_{78} + 4i\frac{n}{\alpha' p^+}\Big)\theta_n^1 &= 0\,,\\
\dot\theta_n^2 + \frac{1}{4}\Big(F_{56}\Gamma_{56} + F_{78}\Gamma_{78} + 4M\Gamma_{5678}\Big)\theta_n^1 - \frac{1}{4}\Big(H_{56}\Gamma_{56} + H_{78}\Gamma_{78} + 4i\frac{n}{\alpha' p^+}\Big)\theta_n^2 &= 0\,.
\end{aligned} \tag{4.14}$$

Differentiating these equations with respect to τ and using them again to eliminate first derivatives gives

$$\ddot\varepsilon_n + m_n\varepsilon_n = 0\,, \tag{4.15}$$

where

$$\begin{aligned}
m_n = \frac{1}{16}\Big\{& F_{56}^2 + F_{78}^2 - 2F_{56}F_{78}\Gamma_{5678} + 16M^2 + +H_{56}^2 + H_{78}^2 - 2H_{56}H_{78}\Gamma_{5678} -\\
&- 8i\frac{n}{\alpha' p^+}(H_{56}\Gamma_{56} + H_{78}\Gamma_{78}) + \frac{16n^2}{(\alpha' p^+)^2}\Big\}\,.
\end{aligned} \tag{4.16}$$

We are following the notation of [46] and we have combined the spinors θ^1 and θ^2 into a single complex spinor $\varepsilon = \theta^1 + i\theta^2$. Considering constant spinors $\varepsilon^{\pm\pm}$ with eigenvalues

$$\begin{aligned}
i\Gamma_{56}\varepsilon^{\pm(\cdot)} &= \pm\varepsilon^{\pm(\cdot)}\,,\\
i\Gamma_{78}\varepsilon^{(\cdot)\pm} &= \pm\varepsilon^{(\cdot)\pm}
\end{aligned} \tag{4.17}$$

under the rotation generators $i\Gamma_{56}$ and $i\Gamma_{78}$ gives

$$\begin{aligned}
m_n\varepsilon^{++} &= \frac{1}{16}\Big\{(F_{56}+F_{78})^2 + 16M^2 + (H_{56}+H_{78})^2 - 8\frac{n}{\alpha' p^+}(H_{56}+H_{78}) + \frac{16n^2}{(\alpha' p^+)^2}\Big\}\varepsilon^{++}\,,\\
m_n\varepsilon^{+-} &= \frac{1}{16}\Big\{(F_{56}-F_{78})^2 + 16M^2 + (H_{56}-H_{78})^2 - 8\frac{n}{\alpha' p^+}(H_{56}-H_{78}) + \frac{16n^2}{(\alpha' p^+)^2}\Big\}\varepsilon^{+-}\,,\\
m_n\varepsilon^{-+} &= \frac{1}{16}\Big\{(F_{56}-F_{78})^2 + 16M^2 + (H_{56}-H_{78})^2 + 8\frac{n}{\alpha' p^+}(H_{56}-H_{78}) + \frac{16n^2}{(\alpha' p^+)^2}\Big\}\varepsilon^{-+}\,,\\
m_n\varepsilon^{--} &= \frac{1}{16}\Big\{(F_{56}+F_{78})^2 + 16M^2 + (H_{56}+H_{78})^2 + 8\frac{n}{\alpha' p^+}(H_{56}+H_{78}) + \frac{16n^2}{(\alpha' p^+)^2}\Big\}\varepsilon^{--}\,.
\end{aligned} \tag{4.18}$$

We know already from the analysis of the bosonic sector that

$$H_{56} = -H_{78} = \alpha_2 \,. \tag{4.19}$$

Hence, at each level n, we find four fermionic oscillators with equal frequencies

$$\omega_n = \sqrt{M^2 + \frac{1}{16}(F_{56} + F_{78})^2 + \frac{n^2}{(\alpha' p^+)^2}}\,, \tag{4.20}$$

which should be associated with the r^i-part of the background and the following two pairs of frequencies[10]

$$\begin{aligned}
\omega_n &= \sqrt{M^2 + \frac{1}{4}\alpha_2^2 + \frac{1}{16}(F_{56} - F_{78})^2 - \alpha_2 \frac{n}{\alpha' p^+} + \frac{n^2}{(\alpha' p^+)^2}}\,, \\
\omega_n &= \sqrt{M^2 + \frac{1}{4}\alpha_2^2 + \frac{1}{16}(F_{56} - F_{78})^2 + \alpha_2 \frac{n}{\alpha' p^+} + \frac{n^2}{(\alpha' p^+)^2}}\,,
\end{aligned} \tag{4.21}$$

which should be associated with the corresponding $(z^a, \bar{z}^a)$ directions of the dual spacetime.

Due to the fact that the background preserves some supersymmetry (as we will verify explicitly in a moment), the above frequencies are expected to be equal to the corresponding bosonic ones. A direct comparison with the results expected from the gauge theory side gives the following two equations for the field strengths of the 3- and 5-form fluxes

$$\begin{aligned}
1 &= M^2 + \frac{1}{16}(F_{56} + F_{78})^2, \\
1 + \alpha_1^2 &= M^2 + \frac{1}{4}\alpha_2^2 + \frac{1}{16}(F_{56} - F_{78})^2 \,.
\end{aligned} \tag{4.22}$$

These constraints also imply the supergravity equations of motion of the next subsection and are not enough to fully determine the unknown constants F_{56}, F_{78} and M. This leaves an abundance of dual backgrounds with the required spectra.[11] As we find in the next subsection, all these backgrounds have the minimal sixteen supersymmetries of a pp-wave, except for a particular one that has eight supernumerary supercharges.

4.3 Spacetime geometry and supersymmetries

The above discussion has led to a type-IIB supergravity background with a non-zero R-R four-form potential C_4 and self-dual field strength and non-zero NS-NS and R-R two-form potentials B_2 and C_2 respectively. The analysis below follows closely the conventions of [46, 47]. As usual, we combine the two-form potentials into a complex potential $A_2 = B_2 + iC_2$ with field strength $G_3 = dA_2 = H_3 + iF_3$ and the self-dual five-form is given by

$$F_5 = \star F_5 = dC_4 - \frac{1}{8}\mathrm{Im}(A_2 \wedge G_3^*)\,, \tag{4.23}$$

[10] Remember that in each of these pairs there should be a ± 1 twist, as required by the gauge theory values of $\Delta - J$, but as noted in the previous subsection, we work here with the rotated coordinate system (2.23), where such twists are not supposed to appear.

[11] Unless we demand maximal supersymmetry the background is not unique even at the $\mathcal{N} = 4$ point, where $a_1 = a_2 = 0$.

JHEP06(2003)015

where * denotes complex conjugation and $\star$ ten-dimensional Hodge duality. The equations of motion are

$$\begin{aligned} R_{ab} &= \frac{1}{6}F_{acdef}F_b^{cdef} + \frac{1}{8}\Big(G_{acd}G_b^{*cd} + G^*_{acd}G_b^{cd} - \frac{1}{6}g_{ab}G_{cde}G^{*cde}\Big), \\ d\star G_3 &= 4iF_5 \wedge G_3\,, \qquad G_{abc}G^{abc} = 0\,, \\ d\star F_5 &= dF_5 = -\frac{1}{8}\mathrm{Im}(G_3 \wedge G_3^*)\,, \end{aligned} \tag{4.24}$$

together with the Bianchi identity $dG_3 = 0$.

In our case, the light-cone worldsheet theory (4.4) and (4.5) leads (after a coordinate rotation of the type (2.23)) to a spacetime metric of the form

$$ds^2 = -4dx^+dx^- - H(x)(dx^+)^2 + \sum_{i=1}^{8} dx^i dx^i\,, \tag{4.25}$$

with

$$H(x) = \sum_{i=1}^{4} x^i x^i + (1+\alpha_1^2)\sum_{i=5}^{8} x^i x^i\,. \tag{4.26}$$

We also get a 5-form flux of the form

$$F_5 = (1+\star)dx^+ \wedge \omega_4\,, \tag{4.27}$$

with

$$\omega_4 = M dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \tag{4.28}$$

and a 3-form flux of the form $G_3 = dx^+ \wedge \xi_2$ with

$$\xi_2 = (\alpha_2 + iF_{56})dx^5 \wedge dx^6 + (-\alpha_2 + iF_{78})dx^7 \wedge dx^8\,. \tag{4.29}$$

The supergravity equations of motion (4.24) give the equation of motion

$$\nabla^2 H = \frac{2}{3}\omega_4^2 + \frac{1}{2}|\xi_2|^2\,, \tag{4.30}$$

where ∇^2 is the laplacian in the transverse eight directions, $\omega_4^2 = \omega_{ijkl}\omega^{ijkl}$ and $|\xi_2|^2 = \xi_{ij}\xi^{*ij}$. Using equations (4.26), (4.28) and (4.29) we get

$$8(2+\alpha_1^2) = 16M^2 + 2\alpha_2^2 + F_{56}^2 + F_{78}^2\,. \tag{4.31}$$

This equation is a consequence of (4.22) and does not imply a new constraint on the 3- and 5-form components. For the special value $F_{56} = -F_{78} = f$ we further get

$$\begin{aligned} f^2 &= 4\alpha_1^2 - \alpha_2^2\,, \\ M^2 &= 1\,. \end{aligned} \tag{4.32}$$

In order for this ansatz to make sense, $4\alpha_1^2$ should be greater than α_2^2. At first order in the deforming parameters, we can explicilty check that this inequality is always satisfied.

Now we would like to find the amount of supersymmetry preserved by the above backgrounds. Our analysis follows closely [19, 46, 48]. Since the background is purely bosonic, the supersymmetry variations of the dilatino and gravitino take the form

$$\begin{aligned}\delta\lambda &= \frac{1}{24} G_{abc}\Gamma^{abc}\epsilon \\ \delta\psi_a &= \mathcal{D}_a\epsilon - \Omega_a\epsilon - \Lambda_a\epsilon^*\end{aligned} \tag{4.33}$$

where[12]

$$\begin{aligned}\Omega_a &= -\frac{i}{480} F_{bcdef}\Gamma^{bcdef}\Gamma_a \\ \Lambda_a &= \frac{1}{96}\left(G_{bcd}\Gamma_a^{\ bcd} - 9G_{abc}\Gamma^{abc}\right),\end{aligned} \tag{4.34}$$

and $\mathcal{D}_a = \partial_a + \frac{1}{4}\omega_{a\widehat{b}\widehat{c}}\Gamma^{\widehat{b}\widehat{c}}$. Note that when necessary, we distinguish tangent space indices from space-time ones by putting a hat on the former.

In terms of the non-coordinate basis

$$e^{\widehat{+}} = dx^+\,, \qquad e^{\widehat{-}} = dx^- + \frac{1}{4}H(x)dx^+\,, \qquad e^i = dx^i\,, \quad i = 1, \ldots, 8 \tag{4.35}$$

for the metric (4.25), the only non-vanishing component of the spin connection reads

$$\omega_{\widehat{+}i} = \frac{1}{2}\partial_i H(x) dx^+\,. \tag{4.36}$$

For the background under consideration, the dilatino variation is given by

$$\delta\lambda = \frac{1}{8}\,\not{\xi}_2\Gamma^{\widehat{+}}\epsilon\,, \tag{4.37}$$

where $\not{\xi}_2 = (\xi_2)_{ij}\Gamma^{ij}$ and it can be written more explicitly as

$$\left(1 - \Gamma^0\Gamma^9\right)\left(\alpha_2\left(\Gamma^{56} - \Gamma^{78}\right) + i\left(F_{56}\Gamma^{56} + F_{78}\Gamma^{78}\right)\right)\epsilon = 0\,. \tag{4.38}$$

We use the standard relations $\Gamma^{\widehat{\pm}} = \frac{1}{2}(\Gamma^0 \pm \Gamma^9)$ and $(\Gamma^i)^2 = (\Gamma^9)^2 = -(\Gamma^0)^2 = 1$.

It is possible to classify the solutions of (4.38) by the eigenvalues of the mutually commuting Lorentz generators $\Gamma^0\Gamma^9, i\Gamma^1\Gamma^2, i\Gamma^3\Gamma^4, i\Gamma^5\Gamma^6, i\Gamma^7\Gamma^8$. Notice that the dilatino variation is independent of the eigenvalues of $i\Gamma^1\Gamma^2, i\Gamma^3\Gamma^4$ but because of the chirality constraint $\Gamma^{11}\epsilon = \epsilon$ we eventually have only a two-fold degeneracy in the number of (complex) solutions.

The standard 16 supersymmetries preserved by generic pp-waves correspond to the spinors with $\Gamma^0\Gamma^9 = 1$, i.e. they are annihilated by $\Gamma^{\widehat{+}}$. We show explicitly in a moment that the gravitino variation is also zero for these. For 3-form R-R fluxes satisfying $F_{56} + F_{78} \neq 0$ we can't obtain any more supersymmetries. For the special case $F_{56} = -F_{78} = f$, however, we get 8 supernumenary Killing spinors with $i\Gamma^5\Gamma^6 = i\Gamma^7\Gamma^8 = \pm$. Hence, this particular string background preserves 24 supersymmetries.

[12] Ω_a should not be confused with the superfield Ω we used earlier.

In order to show that the extra supersymmetries are indeed preserved, we also have to verify that the corresponding gravitino variation vanishes

$$\mathcal{D}_a \epsilon = \Omega_a \epsilon + \Lambda_a \epsilon^* \,. \tag{4.39}$$

For the general background we have[13]

$$\begin{aligned} \Omega_a &= -i\frac{M}{4}(\Gamma^{1234} + \Gamma^{5678})\Gamma^{\hat{+}}\Gamma_a \\ \Lambda_a &= \frac{1}{32}\Big((\xi_2)_{ij}\Gamma_a^{\ \hat{+}ij} - 3G_{abc}\Gamma^{bc}\Big) \,. \end{aligned} \tag{4.40}$$

Since $\Gamma^{\hat{+}}\Gamma_{\hat{-}} = 0$ and $G_{-ab} = 0$, it is easy to see that $\Omega_- = \Lambda_- = 0$. In addition, the component of the spin connection along dx^- is zero. Hence, the Killing spinors are independent of x^- and we can write $\epsilon = \epsilon(x^+, x^i)$.

For the i components, on the other hand, we get

$$\begin{aligned} \Omega_i &= -i\frac{M}{4}(\Gamma^{1234} + \Gamma^{5678})\Gamma^{\hat{+}}\Gamma_i \\ \Lambda_i &= \frac{1}{32}\Big(\Gamma_i \not{\xi}_2 - 8(\xi_2)_{ij}\Gamma_j\Big)\Gamma^{\hat{+}} \,. \end{aligned} \tag{4.41}$$

Using the identity $(\Gamma^{\hat{+}})^2 = 0$ we find that $\Omega_i\Omega_j = \Lambda_i\Lambda_j = \Omega_i\Lambda_j = 0$ and from $\partial_i\epsilon = \Omega_i\epsilon + \Lambda_i\epsilon$ we conclude that $\Omega_i\epsilon$ and $\Lambda_i\epsilon^*$ are x^i-independent and, accordingly, that the Killing spinors take the form $\epsilon = \epsilon_1(x^+) + x^i\epsilon_2(x^+)$. From this we further get $\partial_i\epsilon = \epsilon_2$ and we can eventually set

$$\epsilon = \epsilon_1 + x^i(\Omega_i\epsilon_1 + \Lambda_i\epsilon_1^*) \,. \tag{4.42}$$

Note that the dilatino variation now takes the form $\not{\xi}_2\Gamma^{\hat{+}}\epsilon_1 = 0$.

For the $a = +$ component of (4.39) we have

$$\partial_+\epsilon = \frac{1}{4}\partial_i H(x)\Gamma^i\Gamma^{\hat{+}}\epsilon + \Omega_+\epsilon + \Lambda_+\epsilon^* \tag{4.43}$$

where

$$\begin{aligned} \Omega_+ &= -i\frac{M}{4}(\Gamma^{1234} + \Gamma^{5678})\Gamma^{\hat{+}}\Gamma_{\hat{+}} \\ \Lambda_+ &= -\frac{1}{32}\not{\xi}_2(\frac{1}{2}\Gamma_{\hat{+}}\Gamma_{\hat{-}} + 4) \,. \end{aligned} \tag{4.44}$$

Our background has $H(x) = H_{ij}x^ix^j$ with $H_{ij} = \delta_{ij}$, when $i,j = 1,\dots,4$, $H_{ij} = (1 + a_1^2)\delta_{ij}$, when $i,j = 5,\dots,8$ and zero otherwise. Plugging-in the specific form of the Killing spinors (4.42) into (4.43) and collecting the terms independent of the x^i gives

$$\partial_+\epsilon_1 = -i\frac{M}{2}\left(\Gamma^{1234} + \Gamma^{5678}\right)\epsilon_1 - \frac{1}{32}\not{\xi}_2(\frac{1}{2}\Gamma_{\hat{+}}\Gamma_{\hat{-}} + 4)\epsilon_1^* \,. \tag{4.45}$$

[13] Notice that in our conventions the non-zero components of the tangent space metric read $\eta_{\hat{+}\hat{-}} = -2$, $\eta^{\hat{+}\hat{-}} = -1/2$ and accordingly $\Gamma^{\hat{+}}\Gamma^{\hat{-}} + \Gamma^{\hat{-}}\Gamma^{\hat{+}} = -1$, $\Gamma_{\hat{+}}\Gamma_{\hat{-}} + \Gamma_{\hat{-}}\Gamma_{\hat{+}} = -4$. In addition, $\Gamma_+ = \Gamma_{\hat{+}} + \frac{1}{4}H(x)\Gamma_{\hat{-}}$, $\Gamma_- = \Gamma_{\hat{-}}$, $\Gamma^+ = \Gamma^{\hat{+}}$, $\Gamma^- = \Gamma^{\hat{+}} - \frac{1}{4}H(x)\Gamma^{\hat{+}}$. We frequently use these relations in the ensuing.

We used the equation $(\Gamma^{1234} + \Gamma^{5678})\Gamma^{\hat{-}}\Gamma^{\hat{+}}\epsilon_1 = 0$, which follows from the chirality constraint. Substituting this result into the equation we get from the x^i-linear terms of (4.43) gives

$$\begin{aligned}&\Big(- H_{ij}\Gamma^j\Gamma^{\hat{+}} + M^2\Gamma^i\Gamma^{\hat{+}} + \frac{1}{4}\not{\xi}_2\Lambda_i^* - \frac{1}{8}\Lambda_i\not{\xi}_2^{\,*}\Big)\epsilon_1 + \\ &+\Big(+ iM\{(\Gamma^{1234} + \Gamma^{5678}), \Lambda_i\} - \frac{1}{8}\Omega_i\not{\xi}_2 - \frac{1}{4}\not{\xi}_2\Omega_i\Big)\epsilon_1^* = 0\,. \end{aligned} \tag{4.46}$$

After some Γ-matrix technology we further obtain

$$\left(-H_{ij}\Gamma^j + M^2\Gamma^i + \frac{1}{32}\xi_{jk}\xi^*_{kl}\Gamma_i\Gamma_{jl} + \frac{1}{4}\xi^*_{ij}\xi_{jk}\Gamma_k\right)\Gamma^{\hat{+}}\epsilon_1 - \frac{iM}{8}\Gamma^{1234}\Gamma_i\not{\xi}_2\Gamma^{\hat{+}}\epsilon_1^* = 0\,. \tag{4.47}$$

To derive the above equation we make use of the identities $(\Gamma^{1234} + \Gamma^{5678})\Gamma^{\hat{+}}\epsilon_1 = 0$ and $\Gamma^{1234}\Gamma_i\Gamma^{\hat{+}}\epsilon_1 = +\Gamma^{5678}\Gamma_i\Gamma^{\hat{+}}\epsilon_1$ that hold because of the chirality condition and the relation $[\Gamma_i, \not{\xi}_2] = 4\xi_{ij}\Gamma_j$. In addition, we exploit the fact that our background has non-zero ξ_{ij} components only along the x^5, x^6, x^7, x^8 directions and hence $[\Gamma^{1234}, \not{\xi}_2] = [\Gamma^{5678}, \not{\xi}_2] = 0$.

It is clear from eq. (4.47) that the 16 supersymmetries annihilated by $\Gamma^{\hat{+}}$ are preserved, as expected for generic pp-waves. Moreover, recall that for $F_{56} = -F_{78} = f$ the dilatino variation gave eight extra potential Killing spinors. For this ansatz $\not{\xi}_2 = 2(a_2+if)(\Gamma^{56}-\Gamma^{78})$ and the supernumenary spinors satisfy $(\Gamma^{56}-\Gamma^{78})\epsilon_1 = 0$. As a result of this, $\not{\xi}_2\Gamma^{\hat{+}}\epsilon_1^* = 0$ and the second term of (4.47) vanishes. The first term of (4.47) is also zero. For $i = 1, 2, 3, 4$ it vanishes on the supernumenary spinors provided that $-1+M^2 = 0$, whereas for $i = 5, 6, 7, 8$ it vanishes when $-(1+a_1)^2 + M^2 + \frac{1}{4}|a_2+if|^2 = 0$. Both of these constraints hold, because of the equations of motion (4.32) .

We conclude that the reverse-engineering process that is being employed in this paper does not produce a unique "dual" background at the infinite J limit, but curiously enough, it produces a unique pp-wave string background with the right spectrum and supersymmetry enhancement to 24 supersymmetries. The same process gives a similar abundance of "dual" backgrounds at the $\mathcal{N} = 4$ point as well, but only one of them has maximal supersymmetry and results as the Penrose limit of the full $AdS_5 \times S^5$ geometry.

5. Conclusions and future directions

In this paper we considered the extension of the BMN correspondence for $\mathcal{N} = 1$ superconformal Yang-Mills theories that are obtained as exactly marginal deformations of the $\mathcal{N} = 4$ theory. We concentrated on the gauge theory side of this correspondence and analysed the effect of the deformation on the large R-charge BMN operators of the $\mathcal{N} = 4$ point. First, we noticed that the deforming superpotential breaks the SO(6) R-symmetry group into a U(1), under which all three of the complex Higgs fields are equally charged. For that reason, it was more natural to express the $\mathcal{N} = 4$ BMN correspondence in terms of a "magnetic" pp-wave. With this U(1) we reconsidered the BMN operators both before and after the exactly marginal deformations of the $\mathcal{N} = 4$ theory. On the assumption that such operators are quasi-primary, we used the $\mathcal{N} = 1$ superspace techniques of [17] to derive their anomalous dimensions for finite deforming parameters and verified this result in leading order in perturbation theory.

The picture we find is the following. At the $\mathcal{N}=4$ point we have an infinity of degenerate operators at each level, which are mapped on the string theory side to a corresponding infinite set of Landau-degenerate states. The effect of the deformation on the gauge theory side is to break the $\mathcal{N}=4$ short multiplets into $\mathcal{N}=1$ short and long ones and give anomalous dimensions to many previously protected operators. As a result, the $\mathcal{N}=4$ Landau degeneracy is lifted and we are only left with the three chiral protected operators $\mathrm{Tr}[\Omega^J], \mathrm{Tr}[(\Psi^1)^J]$ and $\mathrm{Tr}[(\Psi^2)^J]$. At the $\mathcal{N}=4$ point we presented the BMN correspondence by using appropriate insertions into the first of these operators only. This works in a natural way at the $\mathcal{N}=4$ point, but less obviously after the deformations, because the Landau degeneracy has been lifted. As for the other previously protected operators with either $\Delta - J = 0$ or $\Delta - J = 2$, we find that they acquire anomalous dimensions which are functions of the deforming parameters.

After the determination of the above anomalous dimensions we ask how we can reproduce them as spectra of an appropriately defined light-cone worldsheet theory. The answer to this question turns out not to be unique, but in all cases we find that the spacetime effect of the deformation is to modify the trace of the transverse metric and turn on 3-form R-R and NS-NS fields. This seems to be consistent with the first and second order analysis of these deformations in the full geometry [6, 12]. It would be interesting to work out explicitly the Penrose limit of these backgrounds and compare with what was obtained in section 4.

Concerning the non-uniqueness of the resulting backgrounds we believe that this is an artifact of the reverse-engineering process at the infinite J limit. It is natural to expect that the extension of our analysis at finite J will produce a unique dual background. It would be interesting to see if this background is the finite J version of the unique pp-wave with twenty-four supernumerary supersymmetries that we find from the infinite J reverse-engineering process.

We consider the analysis of this paper as a first step analysis of the BMN correspondence for conformal $\mathcal{N}=1$ SYM theories. There is an interesting set of subjects that one could further explore. First, it would be nice to understand better how the short and long $\mathrm{SU}(2,2|4)$ representations break up into short and long $\mathrm{SU}(2,2|1)$ representations as we turn on the marginal deformations. It is clear from our analysis that at the $\mathcal{N}=4$ point protected operators belong to certain $\mathcal{N}=1$ short multiplets, which break into long multiplets after the deformation and the previously protected operators become unprotected. This deformation of the multiplets is also evident in the large J version of the equations of motion (3.13). A different effect is the following. The $\mathcal{N}=4$ analysis of [30] focused on single trace BMN operators with two insertions and dimension Δ and showed that they belond in $[\Delta/2]$ long multiplets, thus proving the equality of the anomalous dimensions of several BMN operators. A similar equality is absent after the $\mathcal{N}=4$-breaking deformations and one would like to understand better how these long BMN multiplets rearrange themselves.

Another interesting direction would be to extend our analysis at finite R-charge, as it was done for the $\mathcal{N}=4$ theory in [30, 34]. This will put the correspondence into firmer ground and will allow us to see if and how the reverse-engineering process can produce a unique dual background. It would be nice if such a process, combined with supersymmetry, could lead to the full dual supergravity background of the $\mathcal{N}=1$ theories, which is expected to be a warped fibration of AdS_5 over a deformed S^5 along with 3- and 5-form fluxes. We expect, however, that such a process will be considerably complicated.

Finally, the author of [30] made the interesting observation that the $J=0$ BMN supermultiplet at the $\mathcal{N}=4$ point coincides with the Konishi multiplet and that the large J operators behave like generalized Konishi operators. He also suggested that the BMN classification of operators based on the number of defects could be valid more generally and might provide an alternative route towards a better understanding of the full spectrum of the $\mathcal{N}=4$ theory and the related AdS/CFT correspondence. It seems very possible that the same is true for the $\mathcal{N}=1$ theories in the Leigh-Strassler moduli space and it would be worthwhile to explore this possibility further.

Acknowledgments

We would like to thank G. Dall'Agata, J. Erdmenger, Z. Guralnik, A. Hanany, D. Kutasov, W. McElgin, A. Parnachev, A. C. Petkou, D. Sahakyan and C. Sieg for useful discussions and D. Brecher, D.Z. Freedman, C. V. Johnson and H. Osborn for helpful correspondence. The work of V. N. is supported by the DOE grant DE-FG02-90ER40560. The work of N. P. is supported by the Deutsche Forschungsgemeinschaft under the project number DFG Lu 419/7-2.

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